FINITELY GENERATED PROJECTIVE MODULES
AND T-DENSE IDEALS

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Abstract

Let $C$ be a ring with 1 and $P$ be a finitely generated projective right $C$-module with trace ideal $I$ and $B = \text{End} \ (P_C)$. In this paper we investigate the relation between $N$ and $S$, where $S$ is the intersection of all left ideals of $C$ not containing $I$ and $N$ is the intersection of all $t$-maximal left ideals of $C$.

1. Introduction, Notations and Preliminaries

Throughout this paper $C$ is a ring (with 1), $P$ is a finitely generated projective right $C$-module with trace ideal $I$, $B = \text{End} \ (P_C)$, $J = J(B)$ and $P$ is finitely generated as a left $B$-module.

Given $t$ a left exact radical on $A$-Mod and $M$ in $A$-Mod we say that $M$ is $t$-torsion ($t$-torsion free) provided that $t(M) = M \ (t(M) = 0)$. A submodule $N$ of $M$ is $t$-closed if $M/N$ is $t$-torsion free. A non-zero left $A$-module $M$ is $t$-simple if $M/N$ is $t$-torsion for all submodules $N$ of $M$ such that $N \subsetneq t(M)$. A submodule $N$ of $M$ is $t$-maximal if $M/N$ is $t$-simple and $t$-torsion free. A submodule $N$ of a module $M$ is $t$-dense if $t(M/N) = M/N$.

Define $\text{IM}$ to be the collection of all left ideals $M$ of $C$ such that the trace ideal, $I$, of $P_C$ is not contained in $M$, $S = \bigcap M$, $IK$ is the collection of all $t$-maximal left ideals of $C$, $N = \bigcap K$ and $C(S) = \{c+S \in C/S: c+S \text{ is regular} \}$.

If $X$ is a subset of a ring $R$ then we define the left annihilator of $X$, in $R$, to be $1_{tR} (X) = \{r \in R : r \cdot X = 0 \}$. If $X = \{x\}$ we write $1_{tR} (x)$ for $1_{tR} (\{x\})$. 

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A submodule, $N$, of a module $M$ is said to be small in $M$ if whenever $K$ is a submodule of $M$ such that $K + N = M$, then $K = M$.

In the situation under consideration we have the following facts, the proof of which may be found in (Mohammad, 1987).

**Lemma 1.1:** The following hold:
1. $J(P_C) = PS$
2. $J(BP) = PN$

If in addition to the hypothesis in the situation under consideration we have $B/J$ is semisimple. Then we have the following:

**Proposition 1.2:** The following hold:
i. $C/S$ is semisimple.
ii. $I + S = C$

Recall that for a left ideal $L$ of a ring $C$ and $c \in C$, $Lc^{-1} = \{x \in C : x c \in L\}$.

**Theorem 1.3:** Let $P$ be a finitely generated generator left $B$-module, $C = \text{End } (B^P)$, $\Phi(S) = \{cL \subseteq_C C : Lc^{-1} \cap C(S) \neq \phi, \text{ for all } c \in C\}$ and $\Phi = \{cL \subseteq_C C : I \subseteq L\}$ is the corresponding idempotent topologizing filter, then the following are equivalent:
1. $I + S = C$
2. $\Phi = \Phi(S)$

The following remark is immediate from Theorem 4.3 proved by Ghazi (1983).

**Remark 1.4:** For $P_A$ a flat right $A$-module of type $FP$ and $B = \text{End } (P_A)$ the above mentioned Theorem yields a one-to-one correspondence between the submodules of $P$ (as a left $B$-module) and the $t$-closed left ideals of $A$. In particular, if $X$ is a submodule of $P$ and $D = \{a \in A : Pa \subseteq X\}$ then clearly $D$ is a $t$-closed ideal of $A$ and $X = PD$.

**2. The relation between $N$ and $S$**

In this section we investigation the relation between $N$ and $S$.

The following lemma is needed.
Lemma 2.1: Given any ring $B$ and three left $B$ modules $BX$, $BY$ and $BW$ such that $BY + BX = BW$ with $BX$ minimal with respect to $Y + X = W$, then:

i. $X \cap Y$ is small in $BX$, and

ii. $J(X) = J(W) \cap X$

Proof:

i. Suppose to the contrary that $X \cap Y$ is not small in $X$ then $\exists B X \subsetneq BX$ such that $X \cap Y + X' = X$. Thus $W = Y + X = Y + X \cap Y + X' = Y + X'$, a contradiction to the minimality of $X$. Hence $X \cap Y$ is small in $X$.

ii. Since $X$ imbeds in $W (X \hookrightarrow W)$ then any small submodule of $X$ is mapped to a small submodule of $W$. Thus $J(X) \subseteq J(W) \cap X$. Notice that $W/(X \cap Y) = X/(X \cap Y) \oplus Y/(X \cap Y)$. Now, since the maximal submodules of $W/X \cap Y$ are the maximal submodules of $M$ containing $X \cap Y$ and all the maximal submodules of $M$ contain $X \cap Y (X \cap Y$ is small), then.

$J(W / X \cap Y) = J(W) / (X \cap Y)$. Similarly $J(X/X \cap Y) = J(X) / (X \cap Y)$.

Let $P_1 : W/(X \cap Y) \rightarrow X/(X \cap Y)$ be the projection map, then for $x \in J(W) \cap X$ and $x = x + X \cap Y$ we have $\tilde{x} = x + X \cap Y \in J(X) / (X \cap Y)$, which implies that $x = x_1$ with $x_1 \in J(X)$. Thus $x - x_1 \in X \cap Y$ or $x \in J(X) + X \cap Y = J(X)$.

The following observations are now in place:

1. If $W \in C(S)$, then clearly $Sw \subseteq Cw \cap S$. On the other hand, if $x \in Cw \cap S$ then $x = cw$ and $x \in S$ where $c \in C$. Since $w \in c(S)$ then $c \in S$, thus $x = c w \in Sw$. Thus $Sw = Cw \cap S \forall w \in C(S)$.

2. The same argument above applies to prove that $Sw = Cw \cap N$ for every $w \in C(S)$.

3. Any left ideal of $C$ that contains any $K \in K$ properly has to be an element of $\Phi$, that follows from the definitions.

4. If $Cv = C$ for every $v \in C(S)$ then $S = N$. For if $\exists K \in IK$ such that $S \supsetneq K$, then $K \subsetneq S + K$, thus by (3) above $S + K \in \Phi$. Since $S + K \supseteq S$ also, then $S + K = C = Cv$, thus $1 = s + k$ for some $s \in S$ and $k \in K$ which implies that $S + Ck = C = S + k$. Hence $K \supseteq Ck$ with $k \in C(S)$. Thus $K \supseteq C$, a contradiction.

Theorem 2.2

i. $Nv$ is $t$-dense in $Sv$ if and only if $c(N + 1t_c(v))$ is $t$-dense in $S$. 

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ii. \( Nv \) t-dense in \( Sv \) implies that \( Nv^n \) is t-dense in \( Sv^n \) (\( n \) a positive integer).

**Proof:**

i. Suppose \( Nv \) is t-dense in \( Sv \), then \( Sv/Nv \) is t-torsion. Thus for every \( a \in I \) and \( x \in S \) we have \( axv = yv \) for some \( y \in N \). That is, for every \( a \in I \) and \( x \in S \), \((ax-y)v = 0\), for some \( y \in N \). Thus \( ax-y \in 1_{tc}(v) \), which implies that \( ax \) is an element of \( N + 1_{tc}(v) \). Thus \( S/(N+1_{tc}(v)) \) is t-torsion and hence \( N + 1_{tc}(v) \) is t-dense in \( S \). Conversely, suppose \( N + 1_{tc}(v) \) is t-dense in \( S \). Since \( \rho_v : S \rightarrow Sv \) is a surjective ring homomorphism with \( (N + 1_{tc}(v)) \rho_v = Nv \) then \( \rho_v \) induces \( \hat{\rho}_v : S/(N + 1_{tc}(v)) \rightarrow Sv/Nv \) as a surjective ring homomorphism. Since \( N + 1_{tc}(v) \) is t-dense in \( S \) then \( S/(N + 1_{tc}(v)) \) is t-torsion.

Thus \( Sv/Nv \) is also t-torsion and hence \( Nv \) is t-dense in \( Sv \).

ii. Suppose \( Nv \) is t-dense in \( Sv \). As above \( \rho_v : Sv \rightarrow Sv^2 \) is a surjective ring homomorphism with \( Sv/Nv \) is t-torsion, thus \( Sv^2/Nv^2 \) is t-torsion (\( Nv^2 = \text{Image} \ Nv \) by \( \rho_v \)) and hence \( Nv^2 \) is t-dense in \( Sv^2 \). Now induction on \( n \) completes the proof.

The following two lemmas are needed.

**Lemma 2.3:** For a bisubmodule \( _B X_C \) of the bimodule \( B^P_C \) such that \( X = PH \) for some left ideal \( H \) of \( C \), we have \( X \text{ Hom}(B^P, B_B) \subseteq J = J(B) \Leftrightarrow H \subseteq S. \)

**Proof:** If \( X \text{ Hom}(B^P, B_B) \subseteq J \) then 
\[ X \text{ Hom}(B^P, B_B) P \subseteq JP = PN \text{ with } (X \text{ Hom}(B^P, B_B)) P = X (\text{ Hom}(P_C, C_C)P) = XI = PHI. \]
Thus \( HI \subseteq N \), since \( S + I = C \) then \( HS + HI = H \) with \( HS \) and \( HI \) are subsets of \( S \), thus \( H \subseteq S \).

Conversely, observe first that 
\[
(PS) \text{ Hom}(B^P, B_B) = (PS) (\text{ Hom}(P_C, C_C)P) = (PS) I \subseteq P(SI) \subseteq PN = J(B^P). \]
Thus 
\[
((PS)\text{ Hom}(B^P, BB))P \text{ Hom}(B^P, B_B) = (PS) \text{ Hom}(B^P, B_B)(P \text{ Hom}(B^P, B_B)) = (PS) \text{ Hom}(B^P, B_B) \subseteq (JP) \text{ Hom}(B^P, B_B) = J(P \text{ Hom}(B^P, B_B) = J(B) = J. \]

Now, \( H \subseteq S \Rightarrow PH \subseteq PH \Rightarrow (PH) \text{ Hom}(B^P, B_B) \subseteq (PS) \text{ Hom}(B^P, B_B) \subseteq J. \)
Thus \( X \text{ Hom}(B^P, B_B) \subseteq J. \)
Lemma 2.4: If BF is projective, B any ring and \( x \in \text{Hom}(BF, BB) \subseteq J \) where \( x \in F \), then \( x \in F \cap J(BF) \).

Proof: Let \( BF + BF' \) be free with basis \( \{X_m\}_M = BU \), say. Since \( ug \in J \) for every \( u \in U \) and every \( g \in U \) and every \( g \in \text{Hom}(BU, BB) \) then \( JF \subseteq JU \). Now \( \text{Hom}(BU, BB) \) includes \( \text{Hom}(BF, BB) \times \text{Hom}(BF', BB) \) hence \( x \in \text{Hom}(BF, BB) \times \text{Hom}(BF', BB) \) is \( x \in JF \subseteq JU \). Thus \( x \in JU \cap F = JF \).

Theorem 2.5:

i. For \( v \in C(S) \), if \( PV \) is minimal with respect to the property that \( PV + PS = P \), then \( NV \) is t-dense in \( SV \).

ii. If \( BF = BX \) with \( BF \) projective and \( X \subseteq PS \), then \( \exists v \in C(S) \) such that \( F = PV \) and \( v \) is t-dense in \( SV \).

Proof:

i. \( I + S = C \) implies that \( 1 = s + v \) for some \( s \in C(S) \) and hence \( v + s \) is regular in \( C/S \) which is semisimple. Thus \( v + s \) is a unit which in turn implies that \( CV + S = C \). So, \( mP = P(Cv + S) = PV + PS \).

Since \( PV \) is minimal with respect to \( PV + PS = P \) then by Lemma 2.1 \( J(PV) = J(P) \cap PV \) with \( J(PV) = JPv = PNv \), while \( J(P) \cap PV = Pn \cap PV = Pn \cap PCv = P(N \cap PCv = P(N \cap CV) = PSV \) (by observation 2.). Thus \( PNv = PSV \) and hence \( NV \) is t-dense in \( SV \).

ii. Notice that \( F = PD \) where \( D = \{c \in C : PC \subseteq F\} \). Now \( P(D + S) = PD + PS \) which contains \( F + X = P \). Thus \( P(D + S) = P \), which puts \( D + S \) in \( \phi \). By Theorem 1.3 \( (D+S)c^{-1} \cap C(S) \neq \phi \) for every \( c \in C \), thus \( (D + S) \cap C(S) \neq \phi \). Hence \( D \cap C(S) \neq \phi \). Let \( v \in D \cap C(S) \), then \( CV + S = C \) and hence \( P(CV + S) = P \), so that \( PV + PS = P \) with \( PV \subseteq F \). Thus \( BF = BPv + BF(S \cap PS) \). Since \( \text{Hom}(BF, BB) \subseteq \text{Hom}(BP, BB) \) then \( F \cap PS \) \( \text{Hom}(BF, BB) \subseteq (F \cap PS) \text{Hom}(BP, BB) \) and the latter is contained in \( J \) by lemma 2.3. Thus by lemma 2.4 \( F \cap PS \subseteq JF \) with \( JF \) small forcing \( PV = f \). Now, \( F \cap PS = PV \cap PS = P(CV \cap S) = PSV \subseteq JPV = PNv \). Thus we have \( PSV \subseteq PNV \subseteq PSV \). Thus \( PSV = PNV \) and the proof is complete.
3: Remark:
1. Suppose that \( Nv \) is \( t \)-dense in \( S \) and let \( w \in D(S) = \{ v \in C(S) : Nv \text{ is } t \text{-dense in } S \} \) be such that \( 1tC(w) \) is maximal (among the left annihilators of such \( v \)'s) then \( 1tC(w) = 1tC(w^2) \).

2. If \( B \) is a left Noetherian ring, then \( BP \) is Noetherian. Thus \( C \) has ACC on \( t \)-closed submodules. Since \( 1tC(X) \) is \( t \)-closed (always) then \( 1tC(vn) = 1tC(vn+1) \) for some \( n \in \mathbb{Z}_+ \).

3. We think it is worth mentioning that the case \( I + S \subseteq C \) occurs, and the author produced an example in his Ph.D dissertation (Mohammad, 1987) where he showed that when \( B \) is a field and \( P \) is an infinite dimensional left \( B \)-dimensional left \( B \)-module and \( C = \text{End}_B(P) \) then:
   1. the trace ideal, \( I, \) of \( P_C = \{ c \in C : \text{dim}(Pc) < \infty \} \), and
   2. \( I + S \subseteq C \).

4. The author also proved in his Ph.D dissertation that \( SI \subseteq N \). We point out that if \( IS \subseteq N \) then \( S = N \). The reason for this is that then \( S/N \subseteq C/N \) torsion free while \( S/N \) is torsion, hence \( S/N = 0 \) or \( S = N \).
References


