On Several Kinds of Homogeneous Spaces

All Fora*

Received on Sept. 23, 2000
Accepted for publication on April 17, 2001

Abstract

We shall introduce H-embedded and H-Invariant sets and study some of their properties. We shall discuss the homogeneity of some subspaces as well as the homogeneity of the one-point compactification.

Finally, we introduce a T-Effros property for general topological spaces and we discuss some relations between various kinds of homogeneity including metric spaces having the Effros property.

1. Introduction

Let \((X, \tau)\) be a topological space. The group of all homeomorphisms of \((X, \tau)\) onto itself will be denoted by \(\text{H}(X, \tau)\). \(F_n(X)\), the \(n\)th configuration space of \(X\), will denote the set: \(x^n \cup \{(x_1, \ldots, x_n) \in X^n : x_i = x_j \text{ for some } i \neq j\}\). A space \(X\) is called weakly \(n\)-homogeneous if, given any two points \((x_1, \ldots, x_n)\) and \((y_1, \ldots, y_n)\) of \(F_n(X)\), there exists a homeomorphism \(h\) of \(X\) onto itself such that \(h(x_i) = y_i\) for \(i = 1, \ldots, n\). A space \(X\) is called \(n\)-homogeneous if, given any two points \((x_1, \ldots, x_n)\) and \((y_1, \ldots, y_n)\) of \(F_n(X)\), there exists a homeomorphism \(h\) of \(X\) onto itself such that \(h(x_i) = y_i\) for \(i = 1, \ldots, n\). A space is called homogeneous provided it is 1-homogeneous. A topological transformation group \((G, X)\) is a topological group \(G\) together with a topological space \(X\) and a continuous map \((g, x) \mapsto gx\) of \(G \times X\) into \(X\) such that \((gh)x = g(hx)\). A topological transformation group \((G, X)\) is transitive if for every two points \(x\) and \(y\) of \(X\) there exists \(g \in G\)
such that \( g_x = y \). It is clear that \( X \) is \( n \)-homogeneous if and only if \( H(X) \) acts transitively on \( F_\mu(X) \). A topological space \( (X, \tau) \) is called rigid provided \( H(X) \) is a single point set (i.e. \( H(X) \) consists of the identity mapping only). For \( p \in X \), the homogeneous component \( C_p \) of \( X \) determined by \( p \) is the orbit of \( p \) under the group \( H(X) \), i.e. \( C_p = \{ x \in X : \text{there is an } h \in H(X) \text{ such that } h(x) = p \} \). A topological space \( (X, \tau) \) is called locally homogeneous (abbreviated by LH) at \( p \) in \( X \) if there exists an open set \( U \) in \( X \) containing \( p \) such that, for any \( y \in U \) there is an \( h \in H(X) \) such that \( h(p) = y \). A topological space \( (X, \tau) \) is called locally homogeneous if it is locally homogeneous at each \( p \) in \( X \). It is important to notice that local homogeneity was used in the literature for different meaning (see [1]). It is easy to check that \( X \) is an LH space at \( p \) if and only if \( C_p \) is an open set and, \( X \) is an LH space if and only if each \( C_x \) is clopen (i.e. closed and open simultaneously) in \( X \), \( x \in X \). A metric space \( (X, d) \) is said to have the Effros property if and only if for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for any \( y, z \in X \), if \( d(y, z) < \delta \), then there is an \( h \in H(X) \) such that \( h(y) = z \) and \( d(x, h(x)) < \varepsilon \) for each \( x \in X \).

The following are well known results and will be needed later.

**Theorem 1.1** [3]

If \((X, d)\) is a weakly \( n \)-homogeneous compact connected metric space, then \( X \) is \( n \)-homogeneous or \( X \) is the Euclidean circle.

**Theorem 1.2** [2]

If \((X, d)\) is a compact metric homogeneous space, then it has the Effros property.

For a Tychonoff space \( X \), the Stone-Cech compactification of \( X \) will be denoted by \( \beta X \). For a topological space \( X \), the one-point compactification of \( X \) will be denoted by \( X^* \). If \( A \subseteq \mathbb{R} \) (the set of all real numbers), then \( \tau_u, \tau_{dis}, \tau_{ind}, \tau_{l,r} \) will denote the usual (Euclidean), discrete, indiscrete and the left ray topologies on \( A \), respectively. For the set \( X \), \( |X| \) will denote its cardinality. \( N, Z \) will denote the sets of natural numbers and the integers, respectively.

2. Homogeneity and H-embedded Sets

We start this section by introducing H-embedded sets.
Definition 2.1.

A subset $F$ of $X$ is called an $H$-embedded set in $X$ if for any $g \in H(F)$ there exists $h \in H(X)$ such that $h|F = g$, i.e. every homeomorphism $g : F \to F$ can be extended homeomorphically to a homeomorphism $h : X \to X$.

It is easy to realize that all intervals $(a, b), (a, b], [a, b), [a, b]$ are $H$-embedded sets in the Euclidean space $(\mathbb{R}, \tau_u)$.

The following theorem shows that homogeneous $H$-embedded sets can not lie anywhere in $X$. More precisely, we have the following result.

Theorem 2.2.

Let $F$ be a homogeneous $H$-embedded subset of a space $X$ and let $p \in F$. Then $F \subseteq C_p$, where $C_p$ is the homogeneous component of $X$ that contains $p$.

Proof.

Let $q$ be any element of $F$. Since $F$ is a homogeneous space, there is an $h \in H(F)$ such that $h(p) = q$. But $F$ is an $H$-embedded set in $X$, so there exists $g \in H(X)$ such that $g|F = h$. Consequently, $q \in C_p$.

The following example shows that "homogeneity" condition is not sufficient in Theorem 2.2. Similarly, "$H$-embeddability" condition is not sufficient.

Example 2.3.

(i) By considering the space $([0, 2], \tau_u)$, one can easily notice that $F = \{0, 1\}$ is a homogeneous subspace of $[0, 2]$ but $F \not\subseteq C_{0} = \{0, 2\}$. The reason behind this is that $F$ is not $H$-embedded in $[0, 2]$. Notice that $g : [0, 2] \to F$ defined by $g(x) = x$ is a homeomorphism that can not be extended to an $h$ in $H([0, 2])$.

(ii) By considering the space $([0, 2], \tau_u)$, one can notice that $F = \{0, 2\}$ is an $H$-embedded set in $[0, 2]$ which is not homogeneous. Notice that $F \not\subseteq C_{0} = \{0, 2\}$.

Let $(X, \tau)$ be a topological space and let $A \subseteq X$ and $p \in A$. $C_p^A$ will denote the homogeneous component of $(A, \tau_A)$ that contains $p$ while $C_p$ (as we had mentioned before) is the homogeneous component of $(X, \tau)$ that contains $p$. Some people may probably think that $C_p^A = C_p \cap A$. This equality holds under some restrictions as the following result suggests.
Theorem 2.4.

If $A$ is a homogeneous $H$-embedded set in $(X, \tau)$ and $p \in A$,

then $C^4_A = C_p \cap A$.

Proof:

Using Theorem 2.2, we have $A \subseteq C_p$ and therefore $A = C_p \cap A$. Since $A$ is homogeneous, therefore $C^4_A = A$. Consequently, our equality holds.

The following example shows that the above equality need not hold even if the subspace is open.

Example 2.5.

There exists a homogeneous space $(X, \tau)$ and an open connected subset $A$ such that $C^4_p \neq C_p \cap A$ for some $p \in A$.

To illustrate our example, take $(X, \tau) = (Z, \tau_{i,r})$ and let $A = \{n : n \in N\}$. Here, in fact; $\tau_{i,r} = \{\phi, Z, (-\infty, n) \cap Z : n \in Z\}$. It is easy to observe the following facts: $(X, \tau)$ is a homogeneous space 

$\tau_{i,r, A} = \{\phi, A, (-\infty, n) \cap A : n \in N\}$, $C^4_p = \{p\}$ for all $p \in A$. In fact, 

$(A, \tau_A)$ is a rigid space (this holds, in fact, for all nonempty proper open sets in $X$). It follows that all nonempty proper subsets of $Z$ are $H$-embedded sets in $Z$.

To see this more closely, let $h : (B, \tau_B) \to (B, \tau_B)$ be a homeomorphism, where $B$ is a nonempty proper subset of $Z$. Since $(B, \tau_B)$ is a rigid space, therefore $h$ is the identity mapping on $B$. Now, let $g : (Z, \tau_{i,r}) \to (Z, \tau_{i,r})$ be the identity mapping on $Z$. Then $g \in H(Z)$ and $g|_B = h$. Now, since $Z$ is a homogeneous space, therefore $C_p = Z$ and henceforth $C_p \cap A = A$. However, $C^4_p = \{p\}$. Thus $C^4_p \neq C_p \cap A$.

To state our next result, we need the following definition.

Definition 2.6

A subset $M$ of $X$ is called $H$-invariant if for any $h \in H(X)$, the following holds: if $h(M) \cap M \neq \emptyset$, then $h(M) = M$. 

-258-
An example of $H$-invariant subset is $C_p$ for any $p \in X$. Also, any single point set is $H$-invariant.

**Theorem 2.7.**

Let $(X, \tau)$ be any topological space and let $A \subseteq X$, $p \in A$. Then we have the following:

(i) If $A$ is an $H$-embedded set in $X$, then $C_p^A \subseteq C_p \cap A$.

(ii) If $A$ is an $H$-invariant set in $X$, then $C_p^A \supseteq C_p \cap A$.

**Proof.**

(i) Let $x \in C_p^A$. Then $x \in A$ and, there is a homeomorphism $h : A \to A$ such that $h(p) = x$. Since $A$ is an $H$-embedded set in $X$, there exists a homeomorphism $g : X \to X$ such that $g(t) = h(t)$ for all $t \in A$. Hence $g(p) = h(p) = x$. Consequently, $x \in C_p$. Henceforth $x \in C_p \cap A$.

(ii) Let $y \in C_p \cap A$. Then $y \in A$ and, there exists a homeomorphism $\phi : X \to X$ such that $\phi(p) = y$. Since $y \in A \cap \phi(A)$ and $A$ is an $H$-invariant subset of $X$, therefore $\phi(A) = A$. It follows that

$$\phi |_A : A \to A$$

is a homeomorphism, where $\phi |_A(t) = \phi(t)$, $t \in A$. But since $\phi(p) = y$, $y \in C_p^A$.

**Corollary 2.8.**

Let $(X, \tau)$ be any topological space and let $A$ be any $H$-embedded $H$-invariant subset of $X$. Then $C_p^A = C_p \cap A$ for all $p \in A$.

The following result shows that $H$-embedded sets are preserved under homeomorphisms.

**Theorem 2.9.**

Let $F$ be an $H$-embedded set in $X$ and let $h$ be a homeomorphism from $X$ onto $Y$. Then $h(F)$ is an $H$-embedded set in $Y$.

**Proof.**

Let $E = h(F)$ and let $g : E \to E$ be any homeomorphism. Then $h^{-1} \circ g \circ h : F \to F$ is a homeomorphism from $F$ onto $F$. But $F$ is an $H$-embedded set in $X$, so there is $\phi \in H(X)$ such that $\phi(x) = (h^{-1} \circ g \circ h)(x)$ for all $x \in F$. Now, it is clear that
$h \circ \varphi \circ h^{-1} : Y \to Y$ is a homeomorphism satisfying the condition that $(h \circ \varphi \circ h^{-1})(y) = g(y)$ for all $y \in E$. Consequently, $E$ is an $H$-embedded set in $Y$.

The following result shows that an $H$-embedded set of an $H$-embedded subspace is still an $H$-embedded set in the original space.

**Theorem 2.10.**

Let $A$ be an $H$-embedded set in $B$ and, $B$ be an $H$-embedded set in $X$. Then $A$ is an $H$-embedded set in $X$.

**Proof.**

Let $h \in H(A)$. Since $A$ is $H$-embedded in $B$, there exists $h_1 \in H(B)$ such that $h(t) = h_1(t)$ for all $t \in A$. Also, since $B$ is $H$-embedded in $X$, so there exists $h_2 \in H(X)$ such that $h_2(x) = h_1(x)$ for all $x \in B$. It follows that $h(x) = h_2(x)$ for all $x \in A$. Consequently, $A$ is $H$-embedded in $X$.

The following result exhibits an $H$-embedded set.

**Theorem 2.11.**

Let $(X, \tau)$ be a topological space and let $A$ be a nonempty clopen set in $X$. Then $A$ is an $H$-embedded set in $X$.

**Proof.**

Let $h \in H(A)$. Define $g: X \to X$ as follows: $g(x) = h(x)$ if $x \in A$ and, $g(x) = x$ if $x \in X \setminus A$. Now, it is clear that $g \in H(X)$ and, $g|_A = h$. Hence, $A$ is an $H$-embedded set in $X$.

The following example shows that open (closed) subspaces need not be $H$-embedded sets.

**Example 2.12.**

(i) Let $X = \{(x, y) : x^2 + y^2 = 1 \cup (2, 4) \times (0) \} \subseteq \mathbb{R}^2$ have the usual Euclidean topology. Then $X = H_1 \cup H_2 \cup \{(1,0)\}$, where $H_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \} \setminus \{(1,0)\}$ and $H_2 = (2,4) \times (0)$. Let $\varphi : X \to X$ be a homeomorphism that maps $H_1$ homeomorphically onto $H_2$ and, maps $H_2$ homeomorphically onto $H_1$. Then $\varphi$ cannot be homeomorphically extended to $X$. Hence $H_1 \cup H_2$ is not $H$-embedded in $X$. Notice that $H_1 \cup H_2$ is an open set in $X$.

(ii) Example 2.3 (i) shows that $F$ is a closed non $H$-embedded subspace of $[0,2]$. 

-260-
3. Homogeneity and Subspaces

Regarding locally homogeneous spaces, Example 2.5 shows that \((Z, \tau_{\mathcal{A}^r})\) is a homogeneous (hence locally homogeneous) space in which all of its nonempty proper open subspaces are not locally homogeneous because \(C_{\mathcal{A}} = \{p\}\) is not open in \(\mathcal{A}\). The same Example also shows that a closed subspace of a locally homogeneous space need not be locally homogeneous. However, the following result shows that a subspace of an LH space may be an LH space under certain restrictions.

**Theorem 3.1.**

*If \(A\) is a clopen connected subset of an LH space \(X\), then \(A\) is an LH space.*

**Proof.**

Let \(x \in A\). Since \(X\) is an LH space, there exists an open set \(U\) in \(X\) such that for all \(y \in U\), there is an \(h_y \in \text{H}(X)\) such that \(h_y(x) = y\). Now, for any \(t \in U \cap A\), then \(t \in A\) and \(t \in U\). Hence there is a homeomorphism \(h : X \rightarrow X\) such that \(h_t(x) = t\). Thus \(t \in h_t(A)\) and \(h_t(A)\) is a connected subset of \(X\). The fact that \(t \in A\) forces that \(h_t(A) \subseteq A\). Since \(h_t(A)\) is a nonempty clopen subset of \(A\), therefore \(h_t(A) = A\). Consequently, the mapping \(\varphi : A \rightarrow A\) that is defined by \(\varphi(z) = h_t(z), \ z \in A\), is a homeomorphism satisfying the condition that \(\varphi(x) = h_t(x) = t\). Hence \(A\) is an LH space.

**Theorem 3.2.**

Let \(\{ A_\alpha : \alpha \in \Delta \}\) be a partition for a space \(X\) consisting of clopen sets \(A_\alpha\) in \(X\) such that each \(A_\alpha\) is locally homogeneous in \(X\). Then \(X\) itself is locally homogeneous.

**Proof.**

Let \(p \in X\). Then \(p \in A_\alpha\) for some \(\alpha \in \Delta\). Since \(A_\alpha\) is a locally homogeneous space, so there exists an open set \(U\) in \(A_\alpha\) (hence, it is open in \(X\) because \(A_\alpha\) itself is open in \(X\)) such that for any \(y \in U\), there exists a homeomorphism \(h_y : A_\alpha \rightarrow A_\alpha\) such that \(h_y(y) = p\). Now, we may homeomorphically extend \(h_y\) to \(g_y : X \rightarrow X\) such that \(g_y(A_\alpha) = h_y(A_\alpha)\) by defining \(g_y\) as follows: \(g_y(z) = h_y(z)\) if \(z \in A_\alpha\), and \(g_y(z) = z\) if \(z \in X - A_\alpha\). Hence \(X\) is an LH space.
The following result still clarifies the hereditariness of the LH property with respect to $H$-invariant suspaces.

**Theorem 3.3.**

Let $A$ be a nonempty $H$-invariant subset of an LH space $(X, \tau)$. Then $A$ is an LH space.

**Proof.**

Let $p \in A$. Since $(X, \tau)$ is an LH space, there exists an open set $U$ in $X$ containing $p$ such that for any $q \in U$, there exists $g_q \in H(X)$ such that $g_q(q) = p$. Now, for any $z \in U \cap A$, then $z \in A$ and there exists $g_z \in H(X)$ such that $g_z(z) = p$. Since $p \in g_z(A) \cap A$ and $A$ is an $H$-invariant subspace, therefore $g_z(A) = A$. Consequently, the mapping $\varphi : A \to A$ defined by $\varphi(t) = g_z(t)$ is a homeomorphism satisfying $\varphi(z) = p$. Henceforth $(A, \tau_A)$ is an LH space.

The following example shows that separability is not inherited via homogeneous components.

**Example 3.4.**

(i) Let $X = [0, 1]$ and $\beta = \{ (x, 0) : 0 \leq x \leq 1 \}$. Then $\beta$ is a base for a separable space $(X, \tau(\beta))$. Indeed, $\{0\}$ itself is a dense set in $X$. It is clear that the homogeneous component $C_1 = (0, 1]$ is not separable because it has the discrete relative topology.

(ii) Considering the well known Moore plane $\Gamma$. It is clear that $\Gamma$ is separable and $C_{(0, 0)} = R \times \{0\}$ is not separable because $C_{(0, 0)}$ has the discrete relative topology.

As we have shown that separation axioms are not involved through hereditariness of separability via homogeneous components. The following result can be easily observed. The proof of the following proposition lies in the fact that an open subspace of a separable space is separable.

**Proposition 3.5.**

Let $(X, \tau)$ be a separable LH space at a point $p$ in $X$. Then $C_p$ is a separable space.

The following example shows that normality is not inherited via homogeneous components.
Example 3.6.

Consider the Sorgenfrey plane $S^2$ and let $X = \beta(S^2)$. It is well known that $S^2$ is a first countable Tychonoff homogeneous 0-dimensional nonnormal space. Therefore $C_\rho = S^2$ for all $\rho \in S^2$ (notice that no point in $X \setminus S^2$ is a $G_{\delta}$-set), i.e., $C_\rho$ is not normal even though $X$ is a compact $T_2$ (hence normal) space.

4. Spaces Having the Effros Property

We shall generalize this property for general topological spaces in the following manner.

Definition 4.1.

A topological space $(X, \tau)$ is said to have the $T\text{-Effros property}$ if for any open covering $\mathcal{I}$ for $X$ there exists an open covering $\mathcal{I}'$ for $X$ such that for any $V \in \mathcal{I}'$ and for any $a, b \in V$, there exists an $h \in H(X)$ such that $h(a) = b$ and, for any $x \in X$, $(x, h(x)) \subseteq U$ for some $U \in \mathcal{I}$.

The following result shows that this property (the $T\text{-Effros property}$) is indeed equivalent with the original Effros property in compact metric spaces.

Theorem 4.2.

Let $(X, d)$ be a compact metric space. Then $X$ has the $T\text{-Effros property}$ if and only if $(X, d)$ has the Effros property.

Proof.

$(\Rightarrow)$ Let $X$ have the $T\text{-Effros property}$ and let $\varepsilon > 0$. Take $\mathcal{I} = \{ S_{\varepsilon/2}(p) : p \in X \}$, where $S_r(p)$ denotes the $d$-open ball with center $p$ and radius $r > 0$. Since $X$ has the $T\text{-Effros property}$, there exists an open cover $\mathcal{I}'$ satisfying Definition 4.1. For each $U \in \mathcal{I}'$ and for each $p \in U$ there exists a $\delta(p) > 0$ such that $p \in S_{\delta(p)}(p) \subseteq U$. Since $\mathcal{I}'' = \{ S_{\delta(p)/4}(p) : p \in X \}$ is an open cover of the compact space $X$, so $\mathcal{I}''$ has a finite subcover $\mathcal{I}''' = \{ S_{\delta(p)/4}(p_i) : i = 1, \ldots, n \}$. Let $\delta = \min \{ \delta(p_i)/4 : i = 1, \ldots, n \}$. Now, let $a, b \in X$ such that $d(a, b) < \delta$. Let $j \in \{ 1, \ldots, n \}$ such that $a \in S_{\delta(p_j)/4}(p_j)$. Since $d(a, b) < \delta$, therefore $d(b, p_j) \leq d(a, b) + d(a, p_j) < \delta + \delta(p_j)/4 \leq \delta(p_j)/2$. Hence $a, b \in S_{\delta(p_j)/4}(p_j) \subseteq U$ for some $U \in \mathcal{I}'$. Thus, according to Definition 4.1, there exists an $h \in H(X)$ such that $h(a) = b$ and, for any $x \in X$ we have $(x, h(x)) \subseteq G$ for some $G \in \mathcal{I}$. But this implies that $G = S_{\varepsilon/2}(q)$. 

-263-
for some $q \in X$. Hence $\{ x, h(x) \} \subseteq S_{\varepsilon/2}(q)$ and this yields that 
\( d(x, h(x)) \leq d(x, q) + d(q, h(x)) < \varepsilon \).

(\( \Leftarrow \)) Let \( (X, d) \) have the Effros property and let \( \mathcal{I} \) be any open covering for \( X \). For each \( U \in \mathcal{I} \) and for each \( p \in U \), there exists an \( \varepsilon(p) > 0 \) such that \( p \in S_{\varepsilon(p)/4}(p) \subseteq U \). Since \( \mathcal{I}_i = \{ S_{\varepsilon(p)/4}(p) : U \in \mathcal{I}, p \in U \} \) is an open covering for the compact space \( X \), so \( \mathcal{I}_i \) has a finite subcover \( \mathcal{I}_i' = \{ S_{\varepsilon(p)/4}(p) : i = 1, \ldots, m \} \). Let \( \varepsilon = \min\{ \varepsilon(p_i)/4 : i = 1, \ldots, m \} \). Since \( (X, d) \) has the Effros property, therefore there exists a \( \delta > 0 \) such that for any \( a, b \in X \), if \( d(a, b) < \delta \) then there exists an \( h \in H(X) \) such that \( h(a) = b \) and, for any \( x \in X \) we have \( d(x, h(x)) < \varepsilon \). We claim that \( \mathcal{I}' = \{ S_{\varepsilon/4}(p_i) : i = 1, \ldots, m \} \) satisfies Definition 4.1. To prove our claim, let \( V \in \mathcal{I}' \), i.e. \( V = S_{\delta/4}(p_i) \) for some \( i \in \{ 1, \ldots, m \} \). Let \( a, b \in V \). Then \( d(a, b) \leq d(a, p_i) + d(b, p_i) < \delta/2 < \delta \). Hence there exists an \( h \in H(X) \) such that \( h(a) = b \) and, for any \( x \in X \) we have \( d(x, h(x)) < \varepsilon \). Now, if \( x \in X \); let \( k \in \{ 1, \ldots, m \} \) be such that \( x \in S_{\delta(p_k)/4} \), i.e. \( d(x, p_k) < \varepsilon(p_k)/4 \). Since \( d(x, h(x)) < \varepsilon \leq \varepsilon(p_k)/4 \). Therefore \( d(h(x), p_k) \leq d(h(x), h(x)) + d(x, p_k) < \varepsilon(p_k)/2 < \varepsilon(p_k) \). Hence \( \{ x, h(x) \} \subseteq S_{\delta(p_k)}(p_k) \subseteq U \) for some \( U \in \mathcal{I} \).

As an immediate conclusion of Theorem 1.2 and Theorem 4.2, we have the following result.

Corollary 4.3.

Every compact homogeneous metric space has the T-Effros property.

The following result shows that the T-Effros property is indeed a topological property.

Theorem 4.4.

"The T-Effros property" is a topological one.

Proof.

Let \( X, Y \) be two homeomorphic topological spaces such that \( X \) has the T-Effros property. Then there exists a homeomorphism \( h : Y \to X \). To prove \( Y \) has the T-Effros property, let \( \mathcal{I} \) be any open covering for \( Y \). Then \( \mathcal{I}_i = \{ h(U) : U \in \mathcal{I} \} \) is an open covering for \( X \). Since \( X \) has the T-Effros property, therefore there exists an open covering \( \mathcal{I}'_i \) for \( X \) such that for any \( V \in \mathcal{I}_i \) and for any \( a, b \in V \), there exists \( \varphi \in H(X) \) such that \( \varphi(a) = b \) and for any \( x \in X \), we have \( \{ x, \varphi(x) \} \subseteq U \) for some \( U \in \mathcal{I}_i \). Let \( \mathcal{I}' = \{ h^{-1}(U) : V \in \mathcal{I}'_i \} \). Then \( \mathcal{I}' \) is an open covering for \( Y \). If
On Several Kinds of Homogeneous Spaces

Let \( U \in \mathcal{G}^t \), then \( U = h^{-1}(V) \) for some \( V \in \mathcal{G}^t \). Let \( c, d \in U \). Then \( h(c), h(d) \in V \), so there exists \( \varphi \in H(X) \) such that \( \varphi(h(c)) = h(d) \) and for any \( x \in X \), we have \( \{ x, \varphi(x) \} \subseteq G \) for some \( G \in \mathcal{G}^t \). Now, \( h^{-1} \circ \varphi \circ h \in H(Y) \) satisfies the property that \( h^{-1} (h \circ \varphi \circ h) = d \) and for any \( y \in Y \), then \( \{ y, \varphi(h(y)) \} \subseteq G \) for some \( G \in \mathcal{G}^t \). Thus, there exists \( U \in \mathcal{G} \) such that \( \{ y, \varphi(h(y)) \} \subseteq h(U) \), i.e. \( \{ y, (h^{-1} \circ \varphi \circ h)(y) \} \subseteq U \). Consequently, \( Y \) has the T-Effros property.

A similar result can be extracted for the Effros property. Explicitly, we have the following result.

**Theorem 4.5.**

The Effros property is a topological one.

**Proof.**

It is similar to the preceding proof and it is straightforward.

The following result shows that the T-Effros property is hereditary with respect to some subspaces.

**Theorem 4.6.**

Let \( (X, \tau) \) be a topological space having the T-Effros property and let \( A \) be a nonempty closed \( H \)-invariant subspace. Then \( A \) has the T-Effros property.

**Proof.**

Let \( \mathcal{J} = \{ U_\alpha : \alpha \in \Delta \} \) be any \( \tau \)-open cover for \( A \). For each \( \alpha \in \Delta \), there exists an open set \( V_\alpha \) in \( X \) such that \( U_\alpha = V_\alpha \cap A \). Since \( \mathcal{G}^t = \{ X-A, V_\alpha : \alpha \in \Delta \} \) is an open cover for \( X \) and, \( X \) has the T-Effros property, so there exists an open cover \( \mathcal{J}^t \) for \( X \) satisfying Definition 4.1. Let \( \mathcal{J}^t = \{ H \cap A : H \in \mathcal{G}^t \} \).

Then \( \mathcal{J}^t \) is an open cover for \( A \) and, for any \( a, b \in H \cap A \); where \( H \in \mathcal{G}^t \); there exists a homeomorphism \( \varphi : X \rightarrow X \) such that \( \varphi(a) = b \) and, for any \( x \in X \), we have \( \{ x, \varphi(x) \} \subseteq G \) for some \( G \in \mathcal{G}^t \) (i.e. \( G = V_\alpha \) for some \( \alpha \in \Delta \)). Since \( b \in \varphi(A) \cap A \) and \( A \) is an \( H \)-invariant subspace of \( X \), therefore \( \varphi(A) = A \). Hence the mapping \( g : \mathcal{J}^t \rightarrow A \) defined by \( g(t) = \varphi(t) \); \( t \in A \); is a homeomorphism with the property that for any \( x \in A \), then \( \{ x, g(x) \} = \{ x, \varphi(x) \} \subseteq V_\alpha \cap A = U_\alpha \) for some \( U_\alpha \in \mathcal{J} \).

A similar result holds for the Effros property in metric spaces.
Theorem 4.7.

Let \((X,d)\) be a metric space having the Effros property and let \(A\) be a nonempty \(H\)-invariant subspace of \(X\). Then \((A,d)\) has the Effros property.

For the proof, imitate a similar proof as in Theorem 4.6.

The following result shows that spaces having the T-Effros property as well as the Effros property are stronger than locally homogeneous spaces.

Theorem 4.8.

(i) If \((X,\tau)\) is a topological space having the T-Effros property, then \(X\) is a locally homogeneous space.

(ii) If \((X,d)\) is a metric space having the Effros property, then \((X,\tau(d))\) is an LH space.

Proof.

(i) Let \(\mathcal{I} =\{X\}\). Then there is an open covering \(\mathcal{U}\) for \(X\) satisfying Definition 4.1. Now, for \(a \in X\), let \(V \in \mathcal{U}\) be such that \(a \in V\). Then for any \(b \in V\), there is an \(h \in H(X)\) such that \(h(a) = b\) and, for any \(x \in X\) we have \(\{x, h(x)\} \subseteq X\). Hence \(X\) is an LH space.

(ii) Its proof is similar to (i).

The following result shows that the converse of Theorem 4.8 need not be true.

Example 4.9.

Let \(A = \{(1/2n - 1, 1) : n \in \mathbb{N}, |t| < 1\}\), \(B = \{(1/2n, 0) : n \in \mathbb{N}\}\) and \(X = A \cup B \subseteq \mathbb{R}^2\). Then the usual Euclidean metric space \((X,d_v)\) is an LH metric space because the only homogeneous components of \(X\) are \(A\) and \(B\) and they are both open sets in \(X\). The space \((X,d_v)\) can be easily seen that it does not have the Effros property.

5. Homogeneity and the One Point Compactification

The following example shows that the homogeneities of \(X\) and of its one point compactification \(X^*\) are independent.

Example 5.1.

(i) There exists a homogeneous topological space \(X\) such that \(X^*\) is not homogeneous. Let \(X_1\) be an infinite set and fix \(p \in X_1\). Define a topology \(\tau_1\) on \(X_1\) as follows: \(\tau_1 = \{X_1, A \subseteq X_1 : p \notin A\}\). Then the (excluded point) topology \((X_1, \tau_1)\) is compact and not homogeneous because \(C_p = \{p\}\) while \(C_{\neg p} = X_1 - \{p\}\)
for all \( q \neq p \). Let \( X = X_1 \setminus \{ p \} \). Then \( \tau_X = \tau_{ds} \), so \( (X, \tau_X) \) is a homogeneous metrizable space with its one point compactification \( X^* = X_1 \) not homogeneous.

Another example may be given by \( (-\infty,0), \tau_{t,\infty} \) which is a homogeneous \( T_0 \) connected space but its one point compactification \( (-\infty,0), \tau_{t,\infty} \) is not a homogeneous space.

(ii) Let \( S_1, S_2 \) be two disjoint circles in the Euclidean plane \( R^2 \). Then \((S_1 \cup S_2, \tau_{n})\) is a homogeneous space. Let \( p \in S_1 \) and \( X = S_1 \cup S_2 \setminus \{ p \} \). Then \( X \) is not a homogeneous space even though its one point compactification \( X^* = S_1 \cup S_2 \) is a homogeneous compact metric space.

(iii) There exists a compact homogeneous space \( X \) such that \( X \setminus \{ p \} \) is not homogeneous for all \( p \in X \). Indeed, by taking \( X = \{1,2,3,4\} \) and \( \beta = \{ \{1,2\}, \{3,4\} \} \), then \( (X, \tau(\beta)) \) is the required space.

An obvious result can be obtained through the following.

**Theorem 5.2.**

If \( X \) is an \( n \)-homogeneous space and \( |X| \geq n \), then \( X \setminus \{ p \} \) is \( (n-1) \)-homogeneous for all \( p \in X \).

**Proof.**

Let \( A = \{ x_1, \ldots, x_{n-1} \} \), \( B = \{ y_1, \ldots, y_{n-1} \} \subseteq X \setminus \{ p \} \) such that \( |A| = |B| = n-1 \). Now, \( A \cup \{ p \}, B \cup \{ p \} \) are two \( n \)-element subsets of the \( n \)-homogeneous space \( X \), thus there exists an \( h \in H(X) \) such that \( h(x_i) = y_i, i = 1, \ldots, n-1 \) and \( h(p) = p \). Now, define \( g: X \setminus \{ p \} \rightarrow X \setminus \{ p \} \) by \( g(x) = h(x) \). Then \( g \) is a well-defined homeomorphism satisfying the condition that \( g(x_i) = x_i, i = 1, \ldots, n-1 \).

**Corollary 5.3.**

If \( X^* \); the one point compactification of the space \( X \); is \( n \)-homogeneous, then \( X \) is \( (n-1) \)-homogeneous provided \( |X| \geq n-1 \).

Using Corollary 5.3, it is clear that if \( X^* \) is a 2-homogeneous space then \( X \) is homogeneous.

Using Theorem 1.1, one can obtain the following result.
Theorem 5.4.

If $X^*$ is a weakly 2-homogeneous connected metric space, then $X^*$ is 2-homogeneous or $X^*$ is the Euclidean circle (hence, $X$ is homogeneous).

As an immediate consequence of Theorem 5.4, we have the following result.

Corollary 5.5.

If $X$ is a connected separable metric space such that $X^*$ is weakly 2-homogeneous, then $X$ is homogeneous.

References

