Probabilistic Study of Lateral Earth Pressure on Retaining Walls With Wall Friction

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Received on Jul. 13, 2009 Accepted for publication on Nov. 5, 2009

Abstract

This study is concerned with the use of the Coulomb theory of active and passive lateral earth pressure against retaining walls taking into considerations the variations of both the soil strength represented by the soil friction angle, $\phi$, and the wall friction represented by the wall friction angle, $\delta$. The procedure adopted in this study, basically involves the computation of the active and passive earth pressure coefficients, $k_a$ and $k_p$, and their coefficients of variation, $CV(k_a)$ and $CV(k_p)$. A first order second moment Taylor's series expansion was used to evaluate the variations in these coefficients; the results are presented in graphical forms for a wide range of granular soils. The concept of extreme value distributions (maxima and minima) is utilized in order to evaluate the reliability of the active and passive lateral earth pressure coefficients. Examples to illustrate the design method based on the reliability of these coefficients are presented. Values of load factor utilized in the conventional deterministic approach are correlated with reliability levels.

Keywords: Retaining Walls; Active and Passive Earth Pressure; Seismic Coefficient; Reliability; Probabilistic; Granular Soils.

Introduction

Lateral earth pressure is the most important component in the design of rigid retaining walls, which are widely used in engineering practice. There are several types of rigid retaining walls in common use: 1) cantilever, 2) counterfort or buttressed walls, and 3) gravity. The magnitude of the earth pressure exerted by the soil on the retaining wall mainly depends on the physical properties of the soil, wall friction, the loads conditions and their variations [1]. Due to the large variations encountered in soil properties and loading conditions, probabilistic techniques adopted in this study may provide an alternative design approach for selecting the coefficients of lateral earth pressure as contrasted to the conventional deterministic approach in which the concept of load factor (or factor of safety) is utilized. According to Duncan [2] and Whitman [3] the factor of safety alone is not a sufficient measure for risk assessment and it is hard to evaluate how much safer a retaining wall becomes as the factor of safety increases. Babu and Basha [4] emphasized that the alternative to the conventional approach of using safety factor is
to use probabilistic analysis where uncertainty in the design parameters is considered in a mathematical framework because the main advantage of the probabilistic approach is a direct linkage between uncertainty in the design parameters and probability of failure or reliability. Reliability-based designs and probabilistic approach have been the subject of numerous studies over the past few decades (e.g., [4] – [16]). The general purpose of this study is to suggest a design method based on a predetermined reliability (or probability of failure) for selecting both the active and passive lateral earth pressure coefficients. These coefficients are evaluated using Coulomb Theory that takes into consideration the wall friction [17]. Values of the load factors that can be used in the conventional deterministic approach are obtained at different levels of reliability that takes into account the mean and variance of the independent input variables as well as the distribution function of the dependent variable.

In this study, a first order second moment Taylor’s series expansion was used to evaluate the variations in the lateral earth pressure coefficients and the concept of extreme value distributions (maxima and minima) is utilized in order to evaluate the reliability of these coefficients [18]. The design method based on the reliability of these coefficients is illustrated by two examples. A strong nonlinear relationship between the load factor utilized in the conventional deterministic approach and the reliability level is established.

Mathematical Formulation

Figures 1 and 2 show the forces acting on the soil failure wedge on the retaining wall with a granular backfill for the case of active and passive lateral earth pressures, respectively, taking into account the forces due to wall friction. The active and passive forces per unit length of the wall can, according to Craig [19], be given as follows

$$P_a = \frac{1}{2} \gamma H^2 k_a$$  \hspace{1cm} (1)

$$P_p = \frac{1}{2} \gamma H^2 k_p$$  \hspace{1cm} (2)

where $k_a$ and $k_p$ are the active and passive earth pressure coefficients, respectively, and can be given by the following expressions according to Coulomb Theory [1].

$$k_a = \frac{\sin^2(\alpha + \phi)}{\sin^2(\alpha)\sin(\alpha - \delta)\left(1 + \frac{\sin(\phi + \delta)\sin(\phi - \beta)}{\sin(\alpha - \delta)\sin(\alpha + \beta)}\right)^2}$$  \hspace{1cm} (3)

$$k_p = \frac{\sin^2(\alpha - \phi)}{\sin^2(\alpha)\sin(\alpha + \delta)\left(1 - \frac{\sin(\phi + \delta)\sin(\phi + \beta)}{\sin(\alpha + \delta)\sin(\alpha + \beta)}\right)^2}$$  \hspace{1cm} (4)
where $\phi$ is the soil internal friction angle, $\delta$ is the friction angle between the wall and soil, $\beta$ is the angle of inclination of the backfill behind the wall, $\alpha$ is the angle of inclination of the back face of the wall.

For simplicity, the study presented herein has been performed for $\alpha = 90^\circ$ and $\beta = 0^\circ$. The relationship between the active and passive earth pressure coefficients can be expressed as follows:

$$
\frac{k_p}{k_a} = \left( \frac{\sqrt{\cos \delta + \sin(\phi + \delta)\sin(\phi)}}{\sqrt{\cos \delta - \sin(\phi + \delta)\sin(\phi)}} \right)^2
$$

(5)

Graphical representation of Eq. 5 is shown in Fig. 3 for different values of internal friction angle, $\phi$, and wall friction. It is of interest to point out that, for a given value of friction angle, $\phi$, the passive lateral earth pressure coefficient increases drastically with the increase of the angle of wall friction, $\delta$, without any significant changes in active lateral earth pressure coefficient.

The variations of the coefficients of the active and passive lateral earth pressure with the soil friction angle $\phi$ and wall friction angle $\delta$ are demonstrated in Figs. 4 and 5. Figure 4(a) shows that the $k_a$ value decreases drastically with the soil friction angle $\phi$ for any given value of the wall friction angle $\delta$ whereas Figure 4(b) shows that the $k_a$ value decreases slightly with the wall friction angle $\delta$ for any given value of the soil friction angle $\phi$. Figures 5(a) and 5(b) show that the $k_p$ value increases drastically with the soil friction angle $\phi$ for any given value of the wall friction angle $\delta$ as well as with the wall friction angle $\delta$ for any given value of the soil friction angle $\phi$.

**Design Equations and Charts for Evaluating the Variations of $k_a$ and $k_p$**

In order to assess the variation of the active or passive lateral earth pressure coefficient, $k_a$ or $k_p$ (the dependent variable) utilizing the variations of the design input values of soil friction angle $\phi$ and wall friction angle $\delta$ (the independent variables), a first order second moment Taylor’s series expansion about the mean is used [18]. An explanation of the technique can be illustrated as follows:

Consider the following equation for the dependent variable $Y$ as a function of the independent variables $x_i$:

$$
Y = f(x_1, x_2, \ldots, x_n)
$$

Taking the Taylor series expansion about the mean will yield:

$$
\bar{Y} = f(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)
$$

19
Bars are used over the terms to indicate their mean values and $\sigma^2$ is the variance.

This method of estimating the mean and variance of the random variable has proven to be effective (within 10\%) for actual values, especially when the independent random variables have relatively small coefficients of variation ($CV < 30\%$) and well behaved functions near the mean [16].

The variance of the active (or passive) earth pressure coefficient can be obtained in term of the variation of the values of the mean and variance of $I$ and $G$ using the following expression

$$\sigma_k^2 = \left(\frac{\partial k}{\partial \phi}\right)^2 \sigma_\phi^2 + \left(\frac{\partial k}{\partial \delta}\right)^2 \sigma_\delta^2 \quad (9)$$

Hence, the coefficient of variation of the active or passive earth pressure coefficient may be obtained from the following expression

$$CV(k) = \frac{\sigma_k}{k} \quad (10)$$

Utilizing the mathematical analysis described above, a computer program was established taking into account Eqs. 3 and 4 and their derivatives with respect to $\phi$ and $\delta$ to give design solutions for the conditions mentioned earlier (i.e., $\alpha = 90$, $\beta = 0$). Relatively wide ranges for the internal friction angle of granular (sand) soil (25 to 45 degrees) and for the angle of wall friction (5 to 20 degrees) were considered. The coefficient of variation for both angles was considered to vary from 5 to 40\%.

The results of the computer program for the active and passive earth pressure coefficients are listed in Table 1 for typical $\phi$ and $\delta$ values whereas their coefficients of variation are provided in Table 2 for a specific condition where $\delta = 5^\circ$ and $CV(\delta) = 0.10$ while $\phi$ varies from 25 to 45 degrees and $CV(\phi)$ varies from 5 to 40\%. However, all the results of the coefficients of variation are represented graphically as shown in Figs. 6 through 9 for the active lateral earth pressure and as shown in Figs. 10 through 13 for the passive lateral earth pressure. These graphs can be used in order to evaluate the coefficients of variation of the active and passive lateral earth pressure coefficients as will be demonstrated later in this paper.

**Probabilistic Analysis and Reliability of Earth Pressure Coefficients**

The main goal of any engineering project is to select the most economical and safe design. In the conventional deterministic design approach of selecting a design value for the coefficient of lateral earth pressure, a minimum factor of safety is selected arbitrarily
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Based on past experience with structures that failed or those structures that remained stable under certain loading conditions. These arbitrarily selected factors of safety may be very high resulting in uneconomical designs or may be too low producing unsafe designs. Selecting a minimum factor of safety for a structure may depend on the mean values of input parameters (including the properties of the structure and the loading conditions) as well as on their variations that could in many cases be very high. In such cases probabilistic techniques may provide a better and valuable design method that takes the mean input values and their variations into considerations.

In this study, the following assumptions are made in order to evaluate the design coefficient of the lateral earth pressure acting on retaining walls taking into account the wall friction effects: (A) the design input values are independent random variables, (B) extreme type (I) the active earth pressure coefficient \( k_a \) is assumed to have extreme (maxima) type (I) probability distribution function, and (C) the passive earth pressure coefficient \( k_p \) is assumed to have extreme (minima) type (I) probability distribution function. The extreme type (I) distribution function has been widely used in many structural designs [18].

The reliability \( R \) of a design can be defined as follows:

\[
R = P(X \leq x)
\]  

(11)

In other words, the reliability \( R \) is the probability that the random variable \( X \) is less than or equal to a given selected value \( x \). Alternatively, the probability of failure can be defined as follows:

\[
P_f = 1 - R = 1 - P(X \leq x)
\]  

(12)

It can be observed that if the distribution of the value \( x \) is known (i.e., \( f_x(x) \) is known) then the reliability \( R \) can be determined as shown in Fig. 12.

Since the active earth pressure coefficient \( (k_a) \) is assumed to have type (I) maxima, then the reliability \( R_{k_a} \) of a selected value of \( k_a \) can be given by the following expression

\[
R_{k_a} = F(k_a) = \text{Exp}\{ - \text{Exp}(- \alpha_{k_a} [k_a - u_{k_a}])\}
\]  

(13)

where,

\[
\alpha_{k_a} = \frac{1.282}{\sigma_{k_a}}
\]  

(14)

\[
u_{k_a} = \bar{k}_a - \frac{0.577}{\alpha_{k_a}}
\]  

(15)
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Similarly, the reliability \( R_{kp} \) of a selected value of the passive earth pressure coefficient \( k_p \) may be given by the following expression

\[
R_{kp} = F(k_p) = 1 - \operatorname{Exp}\left\{ - \operatorname{Exp}\left( \alpha_{kp} [k_p - u_{kp}] \right) \right\}
\]  
(16)

where,

\[
\alpha_{kae} = \frac{1.282}{\sigma_{pa}}
\]

(17)

\[
u_{ka} = \bar{k}_p + \frac{0.577}{\alpha_{kp}}
\]

(18)

Normalizing \( k_a \) and \( k_p \) with respect to the mean and standard deviation, and substituting in Eq. 13 and 16 yield the following

\[
R_{ka} = \operatorname{Exp}\left\{ - \operatorname{Exp}\left( - 1.282[Z_{ka} + 0.450] \right) \right\}
\]

(19)

where,

\[
Z_{ka} = \frac{k_a - \bar{k}_a}{\sigma_{ka}}
\]

(20)

\[
R_{kp} = 1 - \operatorname{Exp}\left\{ - \operatorname{Exp}\left( 1.282[Z_{kp} - 0.450] \right) \right\}
\]

(21)

where,

\[
Z_{kp} = \frac{k_p - \bar{k}_p}{\sigma_{kp}}
\]

(22)

The variation of the reliability \( R_{ka} \) is shown graphically in terms of \( Z_{ka} \) in Fig. 15(a) and the variation of the reliability \( R_{kp} \) is shown graphically in terms of \( Z_{kp} \) in Fig. 15(b).

**Examples**

Two examples are presented herein to demonstrate the use of the methodology presented in this study for the following conditions: \( \alpha = 90^\circ \) and \( \beta = 0^\circ \).

**Example (1)**

Consider the following properties for the soil and seismic coefficient \( \phi = 30^\circ \), \( \delta = 5^\circ \), \( CV(\delta) = CV(\phi) = 0.10 \). The design values for \( k_a \) and \( k_p \) are 0.50 and 5.0, respectively. What are the reliabilities of these design values? What is the probability of failure for each case?
From Table 3 (or Figs. 4 and 5), \( \bar{k}_a = 0.3189 \) and \( \bar{k}_p = 3.5052 \)

From Table 2, \( CV(k_a) = 0.1184 \) and \( CV(k_p) = 0.1324 \)

Therefore,

\[
\sigma(k_a) = 0.1184 \times 0.3189 = 0.0378
\]

\[
\sigma(k_p) = 0.1324 \times 3.5052 = 0.4641
\]

If the design value of \( k_a \) is 0.50 and the design value of \( k_p \) is 5.0, then the reliability can be calculated as follows:

\[
R_{ka} = F(k_a)
\]

From Fig. 15(a) (or Eq. 19), \( Z_{ka} = \frac{0.50 - 0.3189}{0.0378} = 4.7910 \), then \( R_{ka} = 0.9988 \).

Therefore, \( P_{f} = 0.0012 \).

Similarly, for \( k_p \)

\[
R_{kp} = F(k_p)
\]

From Fig. 15(b) (or Eq. 21), \( Z_{kp} = \frac{5.0 - 3.5052}{0.4641} = 1.786 \), then \( R_{kp} = 0.9910 \)

Therefore, \( P_{f} = 0.0090 \).

**Example (2)**

Consider the same values for \( \phi = 30^\circ \), \( \delta = 5^\circ \), \( CV(\delta) = CV(\phi) = 0.10 \) as in Example (1) what are the design values of \( k_a \) and \( k_p \) for a reliability of 0.99.

From Fig. 15 (or Eq. 19 and 21), with \( R \) of 0.99: \( Z_{ka} = 3.14 \) and \( Z_{kp} = 1.64 \).

From Example (1), \( \sigma(k_a) = 0.0378 \) and \( \sigma(k_p) = 0.4641 \).

From Eq. 20:

\[
k_a = \bar{k}_a + Z_{ka} \sigma_{ka}
\]

\[
k_a = 0.3189 + 3.14 \times 0.0378 = 0.4376
\]

Similarly, for \( k_p \),

\[
k_p = \bar{k}_p + Z_{kp} \sigma_{kp}
\]

\[
k_a = 3.5052 + 1.64 \times 0.4641 = 4.2663
\]
Based on these calculations of the lateral earth pressure coefficient, the load factor that can be used in the deterministic approach may be obtained by dividing the design value of the lateral earth pressure coefficient by the mean value as follows (for the active case)

$$LF = \frac{k_a}{k_o} = \frac{0.4376}{0.3189} = 1.37$$

Additional values for the load factor $LF$ obtained at different levels of reliability are listed in Table 3. As can be seen from Table 3, there is a strong (nonlinear) relationship between the load factor and the level of reliability selected for the design values; this relationship for the example presented herein is shown in Fig. 16. In other words, the load factor varies depending on the importance of the structure and the variations of the design input values as contrasted to the constant value of load factor used in the conventional deterministic approach.

**Conclusions**

This study investigates the influence of the mean values of the internal friction angle $\phi$ and angle of the wall friction angle $\delta$ and their variations on the active and passive earth pressure coefficients utilizing a first order second moment Taylor’s series expansion. Based on the results of this study, a strong nonlinear relationship between the load factor utilized in the conventional deterministic approach and the level of reliability can be established. The reliability based design approach for selecting the design values for the active and passive earth pressure coefficients are simplified by using graphical form for relatively wide ranges of soil friction angle $\phi$ and wall friction angle $\delta$. 
دراسة إحتمالية لضغط التربة الجانبى على الجدران الإستنادية
مع زاوية الاحتكاك بين التربة والجدار

محمد الزعيبي

ملخص

تهدف هذه الدراسة إلى استخدام نظرية كولوم لضغط التربة الجانبى في حالتية الإيجابية ($k_p$) والسلبية ($k_s$) على الجدران الإستنادية أخذين بعين الاعتبار التغيرات في قوى القصف للتربة والمتمثلة في زاوية إحتكاك التربة ($	heta$). وتتضمن هذه الدراسة حسابات معامل ضغط التربة الجانبي في حالتى الإيجابية والسلبية ومعامل الاختلاف (التشتت) ($CV$) فيهما. فمثلا، فإن استخدم هذه الدراسة طريقة تأليف للعزم لتقييم معامل الاختلاف. تم تلخيص النتائج بشكل رسومات بيانية لتسهل استخدامها. كما تم إعتماد مفهوم الحدود القصوى (الدنيا والعليا) لتقييم المأمونية (Reliability). ولهذا النهاية هذه الدراسة تم استخدام مفهوم توضيحية لإستخدام نتائج هذه الدراسة. وكذلك تم الربط بين مفهوم الطرقية الإعتيادية والتي تستخدم معامل الأمان ومفهوم المأمونية.

References


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Table 1: The coefficients of active and passive earth pressure ($k_a$ and $k_p$).

<table>
<thead>
<tr>
<th>Friction angle, $\phi$</th>
<th>25°</th>
<th>30°</th>
<th>35°</th>
<th>40°</th>
<th>45°</th>
</tr>
</thead>
<tbody>
<tr>
<td>5°</td>
<td>0.3865</td>
<td>2.8335</td>
<td><strong>0.3189</strong></td>
<td><strong>3.5052</strong></td>
<td>0.2604</td>
</tr>
<tr>
<td>10°</td>
<td>0.3726</td>
<td>3.2852</td>
<td>0.3085</td>
<td>4.1433</td>
<td>0.2528</td>
</tr>
<tr>
<td>15°</td>
<td>0.3631</td>
<td>3.8548</td>
<td>0.3014</td>
<td>4.9765</td>
<td>0.2478</td>
</tr>
<tr>
<td>20°</td>
<td>0.3574</td>
<td>4.5968</td>
<td>0.2973</td>
<td>6.1054</td>
<td>0.2450</td>
</tr>
</tbody>
</table>

Table 2: Coefficient of Variation (CV) of the coefficient of active earth pressure $CV(k_a)$ and Coefficient of Variation (CV) of the coefficient of passive earth pressure $CV(k_p)$.

<table>
<thead>
<tr>
<th>Soil friction angle = $\phi^*$</th>
<th>Wall friction angle = $\delta = 5^\circ$</th>
<th>Coefficient of Variation of (DELTA) = 0.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>CV($\phi$)</td>
<td>25°</td>
<td>30°</td>
</tr>
<tr>
<td>0.05</td>
<td>0.0474</td>
<td>0.0542</td>
</tr>
<tr>
<td>0.10</td>
<td>0.0942</td>
<td>0.1049</td>
</tr>
<tr>
<td>0.20</td>
<td>0.1882</td>
<td><strong>0.2080</strong></td>
</tr>
<tr>
<td>0.40</td>
<td>0.3762</td>
<td><strong>0.4152</strong></td>
</tr>
</tbody>
</table>

Bold values are for $CV(k_p)$.

Table 3: Calculated values for Load Factor that can be used in the classical deterministic approach for the active earth pressure coefficient at different levels of reliability.

<table>
<thead>
<tr>
<th>Reliability</th>
<th>Load Factor (LF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.99</td>
<td>1.37</td>
</tr>
<tr>
<td>0.995</td>
<td>1.44</td>
</tr>
<tr>
<td>0.999</td>
<td>1.59</td>
</tr>
<tr>
<td>0.9999</td>
<td>1.80</td>
</tr>
</tbody>
</table>
Figure 1: Forces acting on a retaining wall in the active case (wall moves away from soil)

Figure 2: Forces acting on a retaining wall in the passive case (wall moves toward backfill soil)
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Figure 3: Variations of $k_a$ and $k_p$ for various values of soil friction angle $\phi$ and wall friction angle $\delta$.

Figure 4: (a) Variations of $k_a$ with soil friction angle $\phi$ for various $\delta$-values. (b) Variations of $k_a$ with wall friction angle $\delta$ for various $\phi$-values.
Figure 5: (a) Variations of $k_p$ with soil friction angle $\phi$ for various $\delta$-values. 
(b) Variations of $k_p$ with wall friction angle $\delta$ for various $\phi$-values.

Figure 6: The coefficient of variation of $k_a$ with soil friction angle for various values of coefficient of variation of $\phi$ and wall friction angle $\delta$. $CV(k_a) = 0.05$. 
Figure 7: The coefficient of variation of \( k_a \) with soil friction angle for various values of coefficient of variation of \( \phi \) and wall friction angle \( \delta \): \( CV(k_a) = 0.10 \).

Figure 8: The coefficient of variation of \( k_a \) with soil friction angle for various values of coefficient of variation of \( \phi \) and wall friction angle \( \delta \): \( CV(k_a) = 0.20 \).
Figure 9: The coefficient of variation of $k_a$ with soil friction angle for various values of coefficient of variation of $\phi$ and wall friction angle $\delta$: $CV(k_a) = 0.40$.

Figure 10: The coefficient of variation of $k_p$ with soil friction angle for various values of coefficient of variation of $\phi$ and wall friction angle $\delta$: $CV(k_p) = 0.05$. 
Figure 11: The coefficient of variation of $k_p$ with soil friction angle for various values of coefficient of variation of $\phi$ and wall friction angle $\delta$: $CV(k_p) = 0.10$.

Figure 12: The coefficient of variation of $k_p$ with soil friction angle for various values of coefficient of variation of $\phi$ and wall friction angle $\delta$: $CV(k_p) = 0.20$. 
Figure 13: The coefficient of variation of $k_p$ with soil friction angle for various values of coefficient of variation of $\phi$ and wall friction angle $\delta$: $CV(k_p) = 0.40$.

Figure 14: Design Reliability with known Probability Distribution Function (PDF).
Figure 15: Reliability for active and passive lateral earth pressure coefficients.

Figure 16: Relationship between the load factor (LF) and reliability (R) for the example used in this study (see Table 3).