Saturation Theorem in Simultaneous Approximation by Micchelli Combination of Bernstein Polynomials

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Abstract

The present paper is a continuation of the work in [1,2]. Here we have discussed the saturation result in simultaneous approximation by the iterative combination of Bernstein polynomials, which is introduced by Micchelli [4].

Introduction

Following [1,2], for \( f \in C[0,1] \) was studied direct and inverse theorems in simultaneous approximation by the iterative combination of Bernstein polynomials \( B_n \) introduced by Michelli [4] as:

\[
T_{n,k}(f(u), x) = \left[ I - (I - B_n)^k \right] f(u), x) = \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} B_n^{r} f(u), x
\]

\[
= \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} \int_{0}^{1} W_n(x, \nu) B_n^{r-1} f(u), \nu) d\nu,
\]

where \( B_n^r \) is the \( r \)th iterative (super position) of the operator \( B_n \) and

\[
W_n(x, \nu) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \delta \left( \nu - \frac{k}{n} \right), \delta(t) \text{ being the Dirac-delta function.}
\]

In the present paper, we establish a saturation result in simultaneous approximation by using the iterative combinations of these operators defined in (1.1).

Auxiliary Result

Throughout this paper we assume that \( 0 < a_1 < a_2 < a_3 < b_2 < b_1 < 1 \).

The norm \( \|f\|_{C[a,b]} \) is the sup-norm on \( C[a,b] \), the space of continuous functions on \( [a,b] \). Let \( N \) denote the set of natural numbers, \( N^0 \) the set of nonnegative integers.
By \( \langle a, b \rangle \subset (0, 1) \) denote an open interval containing the closed interval \([a, b]\) and \( C_0 \) denote the set of continuous functions on \((0, 1)\) having a compact support and \( C_0^k \) be the subset of \( C_0 \) of \( k \)-times continuously differentiable function.

**Theorem 2.1** [1]. Let \( f \in C[0, 1] \) and \( k, p \in \mathbb{N} \). If \( f^{(2k+p)} \) exist at some point \( x \in (0, 1) \), then

\[
\lim_{n \to \infty} n^k \left[ T_{n,k}^{(p)}(f,x) - f^{(p)}(x) \right] = \sum_{j=p}^{2k+p} Q(j, k, p, x) f^{(j)}(x),
\]

and

\[
\lim_{n \to \infty} n^k \left[ T_{n,k}^{(p)}(f,x) - f^{(p)}(x) \right] = 0,
\]

where \( Q(j, k, p, x) \) are certain polynomials in \( x \). Further, if \( f^{(2k+p)} \) exists and is continuous on \( \langle a, b \rangle \), then (2.1-2.2) holds uniformly in \([a, b]\).

**Theorem 2.2** [1]. Let \( p \leq q \leq 2k + p \), \( f \in C[0, 1] \) and \( f^{(q)} \) exists and be continuous on \( \langle a, b \rangle \). Then

\[
\left\| T_{n,k}^{(p)}(f, \cdot) - f^{(p)} \right\|_{C[0, 1]} \leq \max \left( C_1 n^{-(q-p)/2} \omega(f^{(q)}, n^{-1/2}), C_2 n^{-(k+1)} \right),
\]

where \( C_1 = C_1(k, p), C_2 = C_2(k, p, f) \) and \( \omega(f^{(q)}, \delta) \) denotes the modulus of continuity of \( f^{(q)} \) on \( (a, b) \).

**Theorem 2.3** [2]. Let \( 0 < \alpha < 2 \) and \( f \in C[0, 1] \). Then in the following statements the implications \((i) \Rightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv)\) hold:

(i) \( f^{(p)} \) exists on \([a_1, b_1]\) and \( \left\| T_{n,k}^{(p)}(f, \cdot) - f^{(p)} \right\|_{C[a_1, a_2]} = O(n^{-\alpha k/2}) \);

(ii) \( f^{(p)} \in \text{Liz}(\alpha, k; a_2, b_2) \);

(iii) (a) \( f^{(m+p)} \) exists and belong to \( \text{Lip}(\alpha k - m; a_2, b_2) \) for \( m < \alpha k < m + 1 \), \( m = 0, 1, 2, \cdots, 2k - 1 \);

(b) \( f^{(m+p)} \) exists and belong to \( \text{Lip}^{*}(1; a_2, b_2) \) for \( \alpha k = m + 1 \), \( m = 0, 1, \cdots, 2k - 2 \);

(iv) \( \left\| T_{n,k}^{(p)}(f, \cdot) - f^{(p)} \right\|_{C[a_1, b_1]} = O(n^{-\alpha k/2}) \).
where \( L_2((\alpha,k; a_2, b_2)) \) denotes the class of continuous functions on \([a_2, b_2]\) for which \( a_{2k}(f, k; a_2, b_2) \leq M h^{\alpha k} \) with some \( M > 0 \). Note that \( L_2((\alpha,1; a_2, b_2)) \) reduces to the Zygmund class \( Lip^*(\alpha, a_2, b_2) \).

**Lemma 2.4** [3]. Let \( 0 < a < b < 1 \). If \( f \in C[0,1] \) and \( g \in C_0 \) with \( \text{supp } g \subset (a,b) \), then

\[
|n^{k+1} \left\{ T_{2n}^{(p)}(f, k, \cdot) - T_n^{(p)}(f, k, \cdot), g(\cdot) \right\}| \leq K \|f\|_{C[0,1]},
\]

where \( K \) is a constant independent of \( f \) and \( n \), and \( \langle h, g \rangle = \int_0^1 h(t)g(t)\,dt \).

**Main Result**

In this section, we shall prove our main result.

**Theorem 3.1** (Saturation Theorem). Let \( f \in C[0,1] \) and \( k, m \in N^0 \) and \( p \in N \). Then, in the following statements, the implications (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) and (iv) \( \Rightarrow \) (v) \( \Rightarrow \) (vi) hold:

(i) \( f^{(p)} \) exists on \([a_1, b_1]\) and \( n^{k+1} \left\| T_{n,k}^{(p)}(f, \cdot) - f^{(p)} \right\|_{C[a_1, b_1]} = O(1) \);

(ii) \( f^{(2k+p+1)} \in A.C.[a_2, b_2] \) and \( f^{(2k+p+2)} \in L_\infty[a_2, b_2] \);

(iii) \( n^{k+1} \left\| T_{n,k}^{(p)}(f, \cdot) - f^{(p)} \right\|_{C[a_1, b_1]} = O(1) \);

(iv) \( f^{(p)} \) exists on \([a_1, b_1]\) and \( n^{k+1} \left\| T_{n,k}^{(p)}(f, \cdot) - f^{(p)} \right\|_{C[a_1, b_1]} = o(1) \);

(v) \( f \in C^{2k+p+2}[a_2, b_2] \) and \( \sum_{j=p}^{2k+p+2} Q(j, k, p, x) f^{(j)}(x) = 0, \quad x \in [a_2, b_2], \)

where \( Q(j, k, p, x) \) are the polynomials occurring in (2.1);

(vi) \( n^{k+1} \left\| T_{n,k}^{(p)}(f, \cdot) - f^{(p)} \right\|_{C[a_1, b_1]} = o(1) \), where the space \( A.C.[a, b] \) is defined as the class of absolutely continuous functions over \([a, b]\) and \( O(1) \) and \( o(1) \)-terms are with respect to \( n \) where \( n \rightarrow \infty \).
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Proof. If \((i)\) holds, following the proof of Theorem 2.3, it is clear that \(f^{(p)}\) is continuous on \([a_1, b_1]\). Also, \((i) \Rightarrow (ii)\) of Theorem 2.3, the derivative \(f^{(2k+p+1)}\) exists and is continuous on \((a_1, b_1)\). Let \(a_1 < a_1' < a_2' < a_2 < b_2 < b_2' < b_1' < b_1\). Then, we can choose a function \(f^*\) with \(\text{supp } f^* \subset (a_1, b_1)\) such that \(f^*(x) = f(x)\) on \([a_1', b_1']\) and that \(f^*\) is \(2k + p + 1\) times continuously differentiable on \((a_1, b_1)\). By Theorem 2.1 and \((i)\) we have

\[
\|T_{n,k}^{(p)}(f^*) - f^{(p)}\|_{C[a_1', b_1']} = O(n^{-k+1}).
\]

Let \(C_0[a_1, b_1]\) denote the set of continuous functions \(q\) on \((0,1)\) with \(\text{supp } q \subset [a_1, b_1]\). Then \(C_0[a_1, b_1]\) is a Banach space with the norm

\[
\|q\| = \max_{x \in [a_1, b_1]} |q(x)|.
\]

Also, let \(C_0^\infty(a_2^*, b_2^*)\) denote the space of infinitely differentiable functions \(g\) on \((0,1)\) with \(\text{supp } g \subset (a_2^*, b_2^*)\). Then there holds

\[
n^{k+1} \left\| T_{n,k}^{(p)} \left( (q, x) - q(x), g(x) \right) \right\| \leq M \|q\|.
\]

It follows that for a \(p\) times continuously differentiable function \(q_*\) and \(\text{supp } q_* \subset [a_1, b_1]\) and for any \(g \in C_0^\infty(a_2^*, b_2^*)\), by (3.3) we get

\[
n^{k+1} \left\| T_{n,k}^{(p)} \left( (q_*, x) - q_*^{(p)}(x), g(x) \right) \right\| = n^{k+1} \left\| T_{n,k} \left( (q_*, x) - q_*(x), g^{(p)}(x) \right) \right\|
\leq M_1 \|q_*\|,
\]

for all \(n\) sufficiently large, where the constant \(M_1\) is independent of \(n\) and \(q_*\).

Since \(f^*\) is continuous on \((a_1, b_1)\), there exists a sequence \(\{f_\sigma\}\) such that \(f_\sigma \in C_0^{2k+p+2}(a_1, b_1)\) and that as \(\sigma \to \infty, f_\sigma \to f^*\) in the sup-norm.
on \([a_1, b_1]\). Then, for any \(g \in C_0^\infty (a_1, b_1)\) and each function \(f_\sigma\), by Theorem 2.1 we get

\[
(3.5) \quad \lim_{n \to \infty} n^{k+1} \left\langle T_{n,k}^{(p)}(f_\sigma, x) - f_\sigma^{(p)}(x), g(x) \right\rangle = \left\langle f_\sigma(x), \sum_{j=1}^{2k+p+2} \mathcal{Q}^*(j, k, p, x) g^{(j)}(x) \right\rangle,
\]

where \(\mathcal{Q}^*_{2k+p+2}(D) = \sum_{j=1}^{2k+p+2} \mathcal{Q}(j, k, p, x) D^j\) denotes the differential operator adjoint to \(\mathcal{Q}_{2k+p+2}(D) = \sum_{j=p}^{2k+p+2} \mathcal{Q}(j, k, p, x) D^j\).

From (3.1) it is clear that there exists a sequence \(\{n_q\}\) of natural numbers such that

\[
(3.6) \quad \lim_{q \to \infty} \left\langle T_{n_q,k}^{(p)}(f^*, x) - f^{*(p)}(x), g(x) \right\rangle = \left\langle h(x), g(x) \right\rangle,
\]

for every \(g \in C_0^\infty (a^*_2, b^*_2)\), where \(h \in L_\infty [a^*_2, b^*_2]\) is a fixed function.

By Lemma 2.4, we have

\[
(3.7) \quad \lim_{q \to \infty} n_q^{k+1} \left| \left\langle T_{n_q,k}^{(p)}(f^* - f_\sigma, x) - (f^* - f_\sigma)^{(p)}(x), g(x) \right\rangle \right| \leq M \|f^* - f_\sigma\|.
\]

Thus, combining (3.5 - 3.7) we get

\[
\left\langle f^*(x), \mathcal{Q}^*_{2k+p+2}(D) g \right\rangle = \lim_{\sigma \to \infty} \left\langle f_\sigma(x), \mathcal{Q}^*_{2k+p+2}(D) g \right\rangle = \left\langle h(x), g(x) \right\rangle.
\]

Hence,

\[
(3.8) \quad \mathcal{Q}_{2k+p+2}(D) f^* = h(x),
\]
as a generalized functions.

Since \(\mathcal{Q}(2k + p + 2, k, p, x) \neq 0\), we can write (3.8) as a first order linear differential equation for \(f^*(2k+p+1)\), it follows that \(f^*(2k+p+1) \in A.C. [a^*_2, b^*_2]\) and also that
\[
f^{*}(2k+p+2) \in L_{\infty}[a_{2}^{*}, b_{2}^{*}]. \text{ From this (ii) is immediate, since } [a_{2}, b_{2}] \subset [a_{2}^{*}, b_{2}^{*}] \text{ and } f^{*} \text{ coincides with } f \text{ on } [a_{2}, b_{2}].
\]

Also, (ii) \(\Rightarrow\) (iii) is immediate from Theorem 2.2.

To prove (iv) \(\Rightarrow\) (v), assuming (iv) and proceeding as in the proof of (i) \(\Rightarrow\) (ii), we obtain

\[
\mathcal{D}_{2k+p+2}(D)f^{*} = 0, \text{ from which (v) is clear.}
\]

Finally, (v) \(\Rightarrow\) (vi) follows from Theorem 2.1.

This completes the proof of the Saturation theorem.

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نظرية التشبع في التقريب المتزامن باستخدام تركيب مايكل لمتعددات برنستن

كريم جبر ثامر

ملخص

أن هذا البحث هو استمرار لما تم برهنته في [1.2]، فكرة البحث هنا هي برهنة نظرية التشبع في التقريب المتزامن باستخدام تركيب مايكل لمتعددات برنستن.
References


