Semi $\omega$-Open Subsets

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Abstract

In this paper we introduce the class of semi $\omega$-open subsets of a topological space $(X, T)$. It is obtained by generalizing $\omega$-open subsets in the same way that semi-open sets had generalized open sets. We carry here a study of the properties of semi $\omega$-open sets, especially in $(X, T)$ and in $(X, T_\omega)$. Finally, we study the semi $\omega$-closure and the semi $\omega$-interior operators.

I. Introduction.

Throughout this work a space will always mean a topological space. If $A$ is a subset of a space $(X, T)$ then $cl(A)$ and $int(A)$ will denote the closure and the interior of $A$, respectively, in $(X, T)$. We usually suppress the $T$ when there is no fear of confusion. We always use $R$ to denote the set of real numbers.

Let $A$ be a subset of a space $(X, T)$. $A$ is called a semi-open subset of $(X, T)$ ([16]) if there exists $U \in T$ such that $U \subseteq A \subseteq cl(U)$. This is equivalent to say that $A \subseteq cl(int(A))$. The family of all semi-open subsets of $(X, T)$ is denoted $SO(X, T)$. The complement of a semi-open set is called a semi-closed set ([16]). The family of all semi-closed subsets of $(X, T)$ is denoted $SF(X, T)$. The set $A$ is called pre-open (or $\beta$-open) ([11]) if $A \subseteq int(cl(A))$. The family of all pre-open subsets of a space $(X, T)$ is denoted $PO(X, T)$. Finally, $A$ is called a semi-preopen subset ([3]) of $(X, T)$ if there exists $P \in PO(X, T)$ such that $P \subseteq A \subseteq cl(P)$. It is clear that

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the definition of a semi-preopen set is obtained by using in the definition of a semi-open set a preopen set \( P \) instead of an open set \( U \).

In Section 2 of the present work, we follow a similar line to introduce semi \( \omega \)-open subsets of a space \((X, T)\) and we study their properties and characterization. In Section 3 we study semi \( \omega \)-open subsets of the spaces \((X, T)\) and \((X, T_\omega)\) and connect them with \(SO(X, T)\) and \(SO(X, T_\omega)\). Finally, in Section 4 we study the semi \( \omega \)-closure and the semi \( \omega \)-interior operators.

II. Semi \( \omega \)-open subsets

Recall that if \( A \) is a subset of a space \((X, T)\) then a point \( x \in X \) is called a condensation point of \( A \) ([4]) if the set \( U \cap A \) is uncountable for each \( U \in T \) with \( x \in U \). In ([5]) a subset \( A \) is called \( \omega \)-closed if \( A \) contains all its condensation points and the complement of an \( \omega \)-closed set is called \( \omega \)-open. The family of all \( \omega \)-open subsets of \((X, T)\), denoted by \( \omega O(X, T) \), forms a topology \( T_\omega \) on \( X \) ([2]). In what follows we generalize \( \omega \)-open sets in the same way that semi-open sets were meant to generalize open sets.

**Definition 2.1.** A subset \( A \) of a space \((X, T)\) is called semi \( \omega \)-open if there exists an \( \omega \)-open set \( U \) of \((X, T)\) such that \( U \subseteq A \subseteq cl(U) \). The family of all semi \( \omega \)-open subsets of \((X, T)\) will be denoted by \( S_\omega O(X, T) \).

Recall that if \( A \) is a subset of a space \((X, T)\) then \( \omega \) int\((A)\) and \( \omega cl(A) \) give the interior of \( A \) and the closure of \( A \), respectively, in the space \((X, T_\omega)\). That Definition 2.1 provides a real generalization of \( \omega \)-open sets is shown in the next example.

**Example 2.2.** We consider the space \((X, T)\) where \( X = \mathbb{R} \) and \( T = \{U \subseteq \mathbb{R} : 1 \notin U \} \cup \{U \subseteq \mathbb{R} : 1 \in U \text{ and } R - U \text{ is finite}\} \). We take the subset \( Q \) of all rational numbers. Since \( U = Q - \{1\} \) is \( \omega \)-open and \( U \subseteq Q \subseteq cl(U) \) then \( Q \in S_\omega O(X, T) \). However, \( Q \notin T_\omega \) because \( \omega \) int\((Q) = Q - \{1\} \).

We immediately have the following characterization of semi \( \omega \)-open sets.

**Theorem 2.3.** A subset \( A \) of a space \((X, T)\) is semi \( \omega \)-open if and only if \( A \subseteq cl(\omega \) int\((A)) \).
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Proof: To prove necessity, let $A \in \omega O(X,T)$. Then there exists $U \in \omega O(X,T)$ such that $U \subseteq A \subseteq \text{cl}(U)$. But $U = \omega \text{int}(U) \subseteq \omega \text{int}(A)$ and thus $\text{cl}(U) \subseteq \text{cl}(\omega \text{int}(A))$.

Hence $A \subseteq \text{cl}(U) \subseteq \text{cl}(\omega \text{int}(A))$. To prove sufficiency, suppose that $A \subseteq \text{cl}(\omega \text{int}(A))$. Put $U = \omega \text{int}(A)$. Then, we have $U \in \omega O(X,T)$ with $U \subseteq A \subseteq \text{cl}(U)$.

The following result is now clear.

Theorem 2.4. For any space $(X,T)$ we have: $T \subseteq T_\omega \subseteq S\omega O(X,T)$.

It follows by example 2.2 that $T_\omega$ is in general a proper subfamily of $S\omega O(X,T)$.

Theorem 2.5. If $\{A_\alpha : \alpha \in I\}$ is a collection of semi $\omega$-open subsets of a space $(X,T)$ then $\bigcup_{\alpha \in I} A_\alpha \in S\omega O(X,T)$.

Proof: For each $\alpha \in I$, there exists $U_\alpha \in \omega O(X,T)$ such that $U_\alpha \subseteq A_\alpha \subseteq \text{cl}(U_\alpha)$. Now $\bigcup_{\alpha \in I} U_\alpha \in \omega O(X,T)$ with $\bigcup_{\alpha \in I} U_\alpha \subseteq \bigcup_{\alpha \in I} A_\alpha \subseteq \bigcup_{\alpha \in I} \text{cl}(U_\alpha) \subseteq \text{cl}(\bigcup_{\alpha \in I} U_\alpha)$. Thus $\bigcup_{\alpha \in I} A_\alpha \in S\omega O(X,T)$.

The intersection of two semi $\omega$-open sets is not in general semi $\omega$-open as it is shown in the next example.

Example 2.6. Again we consider the space $(X,T)$ given in Example 2.2. We take $A = \mathbb{Q}, B = (\mathbb{R} - \mathbb{Q}) \cup \{1\}$. Then $A, B \in S\omega O(X,T)$ while $A \cap B = \{1\} \in S\omega O(X,T)$ since $\omega \text{int}(\{1\}) = \emptyset$.

Theorem 2.7. If $V$ is an open subset of a space $(X,T)$ and $A \in S\omega O(X,T)$ then $V \cap A \in S\omega O(X,T)$.

Proof: Choose $U \in \omega O(X,T)$ such that $U \subseteq A \subseteq \text{cl}(U)$, then $U \cap A \in \omega O(X,T)$ such $U \cap V \subseteq A \cap V \subseteq \text{cl}(U) \cap V \subseteq \text{cl}(U \cap V)$. Therefore $V \cap A \in S\omega O(X,T)$.

Note that Example 2.6 shows that Theorem 2.7 is no longer true if $V$ were taken to be an $\omega$-open subset of $(X,T)$.

Theorem 2.8. Let $(Y,T_Y)$ be a subspace of a space $(X,T)$ and let $A \subseteq Y$. If $A \in S\omega O(X,T)$ then $A \in S\omega O(Y,T_Y)$.
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**Proof** : Let \( A \in \omega O(X, T) \). Then there exists \( U \in \omega O(X, T) \) such that
\[
U \subseteq A \subseteq cl(U) . By \text{Proposition 2.8 of ([2])}, we have \( U \in \omega O(Y, Y_T) \) such that \( U = Y \cap Y \subseteq A \subseteq cl(U) \cap Y = cl_Y(U) \). Thus \( A \in \omega O(X, T) \).

The converse of Theorem 2.8 is false as we show in the next example.

**Example 2.9.** Let \( X = (R, T_U), Y = Q \) and \( A = \{0\} \). Then \( A \in \omega O(Y, Y_T) \) and thus
\[ A \in \omega O(Y, Y_T) \]. But \( A \notin \omega O(R, T_U) \).

**Theorem 2.10.** Let \( (Y, T_Y) \) be a subspace of a space \( (X, T) \) and let \( A \subseteq Y \). If \( Y \in \omega O(X, T) \) and \( A \in \omega O(Y, T_Y) \), then \( A \in \omega O(X, T) \).

**Proof:** Let \( A \in \omega O(Y, T_Y) \). Then there exists an \( \omega \)-open subset \( U \) of \( (Y, T_Y) \) such that \( U \subseteq A \subseteq cl_Y(U) \). But since \( Y \in \omega O(X, T) \) then, by Proposition 2.10 of ([2]), we have \( U \in \omega O(X, T) \) with \( U \subseteq A \subseteq cl_Y(U) \subseteq cl_X(U) \).

Thus \( A \in \omega O(X, T) \).

The next example shows that in Theorem 2.10 the assumption that \( Y \) be \( \omega \)-open can not be weakened to semi \( \omega \)-open.

**Example 2.11.** Here we consider the space \( (X, T) \) given in Example 2.2. Let \( Y = Q \) and \( A = \{0,1\} \). Then \( Y \in \omega O(R, T) \) and \( Y \notin \omega O(R, T) \). Also
\[ A \in \omega O(Y, T_Y) \] but \( A \notin \omega O(R, T) \) since \( cl(\omega int(A)) = cl(0) = \{0\} \).

**Definition 2.12.** A subset \( F \) of a space \( (X, T) \) is called semi \( \omega \)-closed if \( X - F \) is semi \( \omega \)-open. The family of all semi \( \omega \)-closed sets in a space \( (X, T) \) will be denoted by \( \omega F(X, T) \).

Theorems 2.13 - 2.16 are dual to Theorems 2.3, 2.5, 2.4, 2.7 respectively and their proofs are omitted.

**Theorem 2.13.** A subset \( F \) of a space \( (X, T) \) is semi \( \omega \)-closed if and only if \( \text{int}(\omega cl(F)) \subseteq F \).

**Theorem 2.14.** Let \( \{F_\alpha : \alpha \in \Lambda\} \) be a family of semi \( \omega \)-closed subsets of a space \( (X, T) \). Then \( \bigcap_{\alpha \in \Lambda} F_\alpha \) is semi \( \omega \)-closed.
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Theorem 2.15. In any space $(X, T)$ every $\omega$-closed subset of $(X, T)$ is semi $\omega$-closed.

Theorem 2.16. If $B$ is a closed subset of a space $(X, T)$ and $F \in SA_{\omega}(X, T)$, then $F \cup B \in SA_{\omega}(X, T)$.

III. Semi $\omega$-open subsets of $(X, T)$ and $(X, T_\omega)$.

In this section we study relations among the families $SO(X, T), SA_{\omega}(X, T), SO(X, T_\omega)$ and $SA_{\omega}(X, T_\omega)$.

Theorem 3.1. For any space $(X, T)$ we have:

1. $SO(X, T) \subseteq SA_{\omega}(X, T)$.
2. $SF(X, T) \subseteq SA_{\omega}(X, T)$.
3. If $A \in SO(X, T)$ and $A \subseteq B \subseteq cl(A)$ then $B \in SA_{\omega}(X, T)$.

Proof: The easy proof is omitted.

The equality in part (1) of Theorem 3.1 need not hold in general, as it is shown in the next example.

Example 3.2. Consider the space $(R, T_u)$ and its subset $A = R - Q$. Then $A \in SA_{\omega}(R, T_u)$ but $A \not\subseteq SO(R, T_u)$ since $int(A) = \emptyset$.

Theorem 3.3. For any space $(X, T)$ we have:

1. $SA_{\omega}(X, T_\omega) = SO(X, T_\omega)$.
2. $SA_{\omega}(X, T_\omega) \subseteq SA_{\omega}(X, T)$.

Proof: (1) Let $A \in SA_{\omega}(X, T_\omega)$. Then there exists $U \in \omega O(X, T_\omega)$ such that $U \subseteq A \subseteq \omega cl(U)$. But, by Proposition 3.4 of [2]), we have $U \in \omega O(X, T) = T_\omega$. Therefore $U$ is an open set in $(X, T_\omega)$ such that $U \subseteq A \subseteq \omega cl(U)$ and therefore $A \in SO(X, T_\omega)$. The other inclusion follows by Theorem 3.1(1).

(2) Let $A \in SA_{\omega}(X, T_\omega) = SO(X, T_\omega)$. Then there exists $U \in T_\omega$ such that $U \subseteq A \subseteq \omega cl(U)$. But $\omega cl(U) \subseteq cl(U)$ therefore $U \subseteq A \subseteq cl(U)$. Hence $A \in SA_{\omega}(X, T)$.
The equality in part (2) of the above theorem does not hold in general, as we see in Example 2.2, where \( \mathcal{Q} \in \text{SawO}(R,T) \) while \( \mathcal{Q} \not\in \text{SO}(R,T_\omega) \) because the set \( \omega \text{cl}(\omega \text{int}(\mathcal{Q}))=\mathcal{Q}-(1) \) does not contain \( \mathcal{Q} \).

**Theorem 3.4:** For any space \((X,T)\) we have:

1. \( \hat{T} = \text{int} \text{SawO}(X,T) = \{ \text{int}(A) : A \in \text{SawO}(X,T) \} \).
2. \( T_\omega = \omega \text{int} \text{SawO}(X,T) = \{ \omega \text{int}(A) : A \in \text{SawO}(X,T) \} \).

**Proof:** (1) Let \( U \in T \). Then \( U \in \text{SawO}(X,T) \) and since \( U = \text{int}(U), U \in \text{int} \text{SawO}(X,T) \).

Conversely, let \( U \in \text{int} \text{SawO}(X,T) \). Then \( U = \text{int}(A) \) for some \( A \in \text{SawO}(X,T) \) and thus \( U \in T \).

(2) The proof is similar to that of part (1).

**Corollary 3.5.** Let \( T \) and \( M \) be two topologies on a given set \( X \). If \( \text{SawO}(X,T) \subseteq \text{SawO}(X,M) \) then \( T \subseteq M \) and \( T_\omega \subseteq M_\omega \).

**Proof:** By Theorem 3.4 (1), we have \( T = \text{int} \text{SawO}(X,T) \subseteq \text{int} \text{SawO}(X,M) = M \). That \( T_\omega \subseteq M_\omega \) follows similarly by Theorem 3.4 (2).

**Corollary 3.6.** Let \( T \) and \( M \) be two topologies on a given set \( X \). If \( \text{SawO}(X,T) = \text{SawO}(X,M) \) then \( T = M \) and \( T_\omega = M_\omega \).

Note that the converse of Corollary 3.5, is false, as shown by the next example.

**Example 3.7.** Let \( X = R \) and let \( T_u \) and \( T_s \) be the usual topology and the Sorgenfrey line topology on \( R \), respectively. Then \( T_u \subseteq T_s \), but \( A = [0,1] \in \text{SawO}(R,T_u) \) and \( A \notin \text{SawO}(R,T_s) \).

Recall that a space \((X,T)\) is called anti-locally countable ([2]) if each non-empty open subset of \((X,T)\) is uncountable.

**Theorem 3.8.** Let \((X,T)\) be an anti-locally countable space. If \( A \) is \( \omega \)-closed and \( A \in \text{SawO}(X,T) \), then \( A \in \text{SO}(X,T) \).

**Proof:** Let \( A \) be an \( \omega \)-closed subset of \( X \) such that \( A \in \text{SawO}(X,T) \). Then \( A \subseteq \text{cl}(\omega \text{int}(A)) \). By Corollary 3.14 of ([2]), we have \( \omega \text{int}(A) = \text{int}(A) \) and therefore \( A \subseteq \text{cl}(\text{int}(A)) \). Thus \( A \in \text{SO}(X,T) \).
Example 3.9. To show that in theorem 3.8 the assumption on \((X, T)\) to be anti-
locally countable is essential, we consider the space \((X, T)\) where
\(X = \{a, b, c\}\) and \(T = (\emptyset, X, \{a\}, \{b, c\})\). We take \(A = \{a, b\}\). Then
\(A \in S\omega O(X, T)\) while \(A \notin SO(X, T)\) since \(cl(int(A)) = cl(a) = \{a\}\).

Also, in Theorem 3.8 it is essential that the set \(A\) be \(\omega\)-closed, as can be seen by
Example 3.2.

Theorem 3.10 Let \((X, T)\) be an anti-locally countable space. If \(A\) is \(\omega\)-open and
\(A \in S\omega F(X, T)\), then \(A \in SF(X, T)\).

Proof: Follows similarly by using Proposition 3.12 of ([2]).

IV. Semi \(\omega\)-closure and Semi \(\omega\)-Interior Operators.

We start by the following definition.

Definition 4.1. Let \(A\) be a subset of a space \((X, T)\). Then the semi \(\omega\)-closure of
\(A\), denoted \(S\omega cl(A)\), is defined as the intersection of all semi \(\omega\)-closed
subsets of \((X, T)\) containing \(A\). We observe, by Theorem 2.14, that
\(S\omega cl(A) \in S\omega F(X, T)\).

Theorem 4.2. If \(A\) is a subset of a space \((X, T)\), then \(S\omega cl(A) = A \cup int(\omega cl(A))\).

Proof: We observe that
\(int(\omega cl(A) \cup int(\omega cl(A))) \subseteq int(\omega cl(A) \cup \omega cl(A)) = int(\omega cl(A)) \subseteq A \cup int(\omega cl(A))\).
Thus, by Theorem 2.13, we conclude that \(A \cup int(\omega cl(A))\) is a semi \(\omega\)-closed
subset of \((X, T)\) containing \(A\) and so \(S\omega cl(A) \subseteq A \cup int(\omega cl(A))\). On the
other hand, since \(S\omega cl(A) \in S\omega F(X, T)\), then
\(int(\omega cl(S\omega cl(A))) \subseteq S\omega cl(A)\). Therefore
\(int(\omega cl(A)) \subseteq int(\omega cl(S\omega cl(A))) \subseteq S\omega cl(A)\) and consequently
\(A \cup int(\omega cl(A)) \subseteq S\omega cl(A)\). Thus \(S\omega cl(A) = A \cup int(\omega cl(A))\).

Recall that if \(A\) is a subset of a space \((X, T)\) then \(S int(A)\) and \(Scl(A)\) give
the semi-interior of \(A\) and the semi-closure of \(A\), respectively, in the space
\((X, T)\) ([3]).

Corollary 4.3. Let \(A\) be a subset of a space \((X, T)\). Then
\(S\omega cl(A) \subseteq Scl(A) \cap \omega cl(A)\).

Proof: \(S\omega cl(A) = A \cup \omega cl(A) \subseteq \omega cl(A)\) (by Theorem 1.5 of ([3])). Also,
\(S\omega cl(A) \subseteq A \cup \omega cl(A) = \omega cl(A)\). Therefore, \(S\omega cl(A) \subseteq Scl(A) \cap \omega cl(A)\).
The equality in Corollary 4.3 does not hold in general as we show in the next example.

Example 4.4. Consider the space \( X = (\mathbb{R}, T_\omega) \) and let \( A = \mathbb{Q} \cup (0,1) \). Then

\[
\text{scl}(A) = \mathbb{R} \quad \text{and} \quad \text{awl}(A) = \mathbb{Q} \cup [0,1].
\]

On the other hand

\[
\text{sawl}(A) = A \cup \text{int}(\text{awl}(A)) = A \cup (0,1) = A.
\]

Definition 4.5. If \( A \) is a subset of a space \((X,T)\), then the semi \( \omega \)-interior of \( A \), denoted \( \text{s}\omega \text{ int}(A) \), is the union of all semi \( \omega \)-open subsets of \((X,T)\) contained in \( A \). We observe that, by Theorem 2.5, \( \text{s}\omega \text{ int}(A) \in \text{Sow}(X,T) \).

Theorem 4.6. Let \( A \) be a subset of a space \((X,T)\). Then

\[
\text{s}\omega \text{ int}(A) = A \cap \text{cl}(\omega \text{ int}(A)).
\]

Proof: We observe

\[
A \cap \text{cl}(\omega \text{ int}(A)) \subseteq \text{cl}(\omega \text{ int}(A \cap \omega \text{ int}(A))) \subseteq \text{cl}(\omega \text{ int}(A \cap \text{cl}(\omega \text{ int}(A)))).
\]

So

\[
A \cap \text{cl}(\omega \text{ int}(A)) \text{ is, by Theorem 2.3, a semi } \omega \text{-open set contained in } A, \text{ and thus } A \cap \text{cl}(\omega \text{ int}(A)) \subseteq \text{s}\omega \text{ int}(A). \text{ On the other hand,}
\]

since \( \text{s}\omega \text{ int}(A) \in \text{Sow}(X,T) \), we have

\[
\text{s}\omega \text{ int}(A) \subseteq \text{cl}(\omega \text{ int}(\text{s}\omega \text{ int}(A))) \subseteq \text{cl}(\omega \text{ int}(A)).
\]

Thus

\[
\text{s}\omega \text{ int}(A) = A \cap \text{cl}(\omega \text{ int}(A)).
\]

Corollary 4.7. Let \((X,T)\) be a space and let \( A \subseteq X \). Then:

(a) \( A \in \text{Sow}(X,T) \) if and only if \( \text{s}\omega \text{ int}(A) = A \).

(b) \( A \in \text{Sow}(X,T) \) if and only if \( \text{sawl}(A) = A \).

(c) \( \text{s}\omega \text{ int}(X - A) = X - \text{sawl}(A). \)

(d) \( \text{sawl}(X - A) = X - \text{s}\omega \text{ int}(A). \)

Theorem 4.8. Let \((X,T)\) be a space and \( D \subseteq X \). Then the following conditions are equivalent:

(1) \( D \) is a dense subset of \((X,T_\omega)\).

(2) \( \text{sawl}(D) = X \).

(3) If \( F \in \text{Sow}(X,T) \) and \( D \subseteq F \), then \( F = X \).

(4) For every non-empty semi \( \omega \)-open subset \( A \) of \((X,T)\), \( A \cap D \neq \emptyset \).

(5) \( \text{s}\omega \text{ int}(X - D) = \emptyset \).
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Proof: The easy proof is left to the reader.

Theorem 4.9. Let \((X, T)\) be anti-locally countable space and let \(A \subseteq X\). Then
\[ sω \text{int}(sωcl(A)) = sωcl(A) \cap cl(ω \text{int}(ωcl(A))). \]

Proof: We have
\[ sω \text{int}(sωcl(A)) = sωcl(A) \cap cl(ω \text{int}(sωcl(A))) \subseteq sωcl(A) \cap cl(ω \text{int}(ωcl(A))) = sωcl(A) \cap cl(ωcl(A)). \]
Where the last inequality follows by \([2]\), because \(ωcl(A)\) is an \(ω\)-closed subset of anti-locally space \((X, T)\). On the other hand we have:
\[ sω \text{int}(sωcl(A)) = sωcl(A) \cap cl(ω \text{int}(A) \cup \text{int}(ωcl(A))) \]
\[ \supseteq sωcl(A) \cap cl(ω \text{int}(A) \cup \text{int}(ωcl(A))) = sωcl(A) \cap cl(\text{int}(ωcl(A))). \]

In the above theorem it is essential for \((X, T)\) to be anti-locally countable as we show in the next example.

Example 4.10. In the space \((X, T)\) of Example 3.9 we take the subset \(A = \{a, b\}\).

Then \(sωcl(A) = A \cup \text{int}(ωcl(A)) = A \cup \text{int}(A) = A\).
\[ sω \text{int}(A) = A \cap cl(ω \text{int}(A)) = A \cap cl(A) = A \text{ but} \]
\[ cl(\text{int}(ωcl(A))) = cl(\text{int}(A)) = cl(\{a\}) = \{a\} \text{ and so} \]
\[ sωcl(A) \cap cl(\text{int}(ωcl(A))) = \{a\} = sω \text{int}(sωcl(A)) = A. \]

Theorem 4.11: Let \((X, T)\) be anti-locally countable space and let \(A \subseteq X\). Then
\[ sωcl(sω \text{int}(A)) = sω \text{int}(A) \cup \text{int}(ω \text{int}(A))). \]

Proof: The proof is similar to the proof of Theorem 4.9 by using Theorem 4.6 and Proposition 3.12 \([2]\).
References


