The Approximation of the Geometric Distribution by Gamma Distribution

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Abstract

The geometric distribution has a property similar to that of the non-aging or "Markovian" property of the exponential distribution

\[ P(X = x + k / X \geq k) = P(X = x) \]

Also the geometric distribution is a discrete analogue of the exponential distribution.

My work is to find a suitable continuous exponential distribution to approximate the geometric distribution.

Introduction:

A geometric progression is a sequence of values each of which after the first is obtained by multiplying the preceding one by a constant value called the common ratio.

Example: \( e^{-x}, e^{-2x}, e^{-3x}, \ldots \) with \( e^{-x} \) as a common ratio

The sum of these values

\[ S = \sum_{i=1}^{\infty} e^{-xi} = \frac{e^{-x}}{1 - e^{-x}} \]

The sequence of values: \( p, pq, pq^2, \ldots \) is also a geometric with a common ratio \( q \).

Let \( T = \sum_{x=1}^{\infty} pq^{x-1} = \frac{p}{1 - q} \)

If \( T = S \) term wise then \( p = e^{-x}, pq = e^{-2x}, \ldots \Rightarrow p = e^{-x} = q \Rightarrow p = q \)

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So the best approximation of $S$ by $T$ when $p = q$ and $p + q = 1$ $\Rightarrow$ $p = \frac{1}{2}$

Let $X$ be a geometric random variable with probability of success $p$.

$X: G(p)$

$P(X = x) = g(x) = pq^{x-1}$ $x = 1, 2, 3, \ldots$

What are the possible values of $p$ to approximate the geometric distribution by the Gamma distribution?

The following table gives us some indications where we take $p = \frac{1}{\alpha}$ $\alpha = 1, 2, 3, \ldots$

as the probability of success for the geometric distribution.

Let us have the two probability densities for Gamma and geometric, respectively:

$h(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}$ $x > 0,$

$g(x) = \frac{1}{\alpha} \left( \frac{x-1}{\alpha} \right)^{\alpha-1}$ $x = 1, 2, 3, \ldots$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$E(x) = \alpha \beta$</th>
<th>$\sigma_x^2 = \alpha \beta^2$</th>
<th>$\frac{E(x)}{p} = \frac{1}{p}$</th>
<th>$\sigma_x^2 = \frac{q}{p^2}$</th>
<th>The mode of $x_\alpha$ at $x = 1$</th>
<th>The mode of $x_\alpha$ at $x = 1$</th>
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<td>$\alpha(\alpha - 1)$</td>
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The approximation of the geometric distribution by Gamma distribution

From this table we can conclude that the family of Gamma distributions for $\beta = 1$ only, and the family of geometric distributions have the same mean $\frac{1}{p} = \alpha$ but their variances are proportional such that $\sigma^2_x = (\alpha - 1)\sigma^2_\alpha$, $\forall \alpha \geq 3$

But the two distributions are very close together at $\alpha = 2$

References