Centers of Groups Acting on Trees

Rasheed Mahmood*

Received on 27/5/1996 Accepted for publication on 30/6/1997

Abstract

In this paper we prove that if $G$ is a group acting on a tree $X$, $T$ is a tree of representatives and $Y$ is a fundamental domain for the action of $G$ on $X$ such that $T \subseteq Y$ and $G_x \neq G_{t(x)}$ for all $x \in E(X)$, then the center $C(G)$ of $G$ is given by

$$C(G) = \bigcap_{y \in \overline{Y}} C(G_y) \cap \bigcap_{y \in E(Y)} \overline{G_y}$$

where $C(G_y)$ is the center of $G_y$ and $\overline{G_y}$ is a certain subgroup of $G_y$.

1. Introduction and preliminaries

It is well known that free groups of rank greater than one and non-trivial free products of groups have trivial centers. Magnus, Karrass and Solitar (1966) showed that if $G$ is the free product of the groups $A$ and $B$ with amalgamation subgroup $H$ such that $H \neq A$ and $H \neq B$ then the center $C(G)$ of $G$ is given by

$$C(G) = C(A) \cap C(B) \cap H$$

where $C(A)$ and $C(B)$ are the centers of $A$ and $B$ respectively.

Free groups and free product of groups with amalgamation subgroup are special cases of groups acting on trees.

In this paper we generalize the above result to groups acting on trees to include tree product of groups and HNN groups.

We begin by giving preliminary definitions. By a graph $X$ we understand a pair of disjoint sets $V(X)$ and $E(X)$ with $V(X)$ non-empty together with a map $E(X) \rightarrow V(X) \times V(X)$, $y \rightarrow (o(y), t(y))$, and a mapping $E(X) \rightarrow E(X)$, $y \rightarrow \overline{y}$ satisfying $\overline{\overline{y}} = y$ and $o(\overline{y}) = t(y)$ for all $y \in E(X)$. The case $\overline{y} = y$ is possible for some $y \in E(X)$. For $y \in E(X)$, $o(y)$ and $t(y)$ are called the ends of $y$ and...
\( \tilde{y} \) is called the inverse of \( y \). For definitions of trees, groups acting on trees, etc, we refer the readers to Serre(1977) or Mahmud(1989, 1991). Let \( G \) be a group acting on a tree \( X \), \( T \) and \( Y \) be two subtrees of \( X \). Then \( T \) is called a tree of representatives if \( T \) contains exactly one vertex from each \( G \)-vertex orbit and \( Y \) is called a fundamental domain for the action of \( G \) on \( X \) if \( Y \) contains a tree of representatives \( T_0 \) (say) such that each edge of \( Y \) has at least one end in \( T_0 \), and \( Y \) contains exactly one edge \( y \) from each \( G \)-edge orbit if \( y \) and \( \tilde{y} \) are not in the same \( G \)-edge orbit, and exactly one pair \( x \) and \( \tilde{x} \) from each \( G \)-edge orbit if \( x \) and \( \tilde{x} \) are in the same \( G \)-edge orbit.

For the rest of this section \( G \), \( X \), \( T \), and \( Y \) will be as above such that \( T \subseteq Y \).

We have the following notation.

(i) For each \( x, y \in X \), define \( G(x, y) \) to be the set \( G(x, y) = \{ g \in G : g(x) = y \} \) and \( G(x, x) = G_x \), the stabilizer of \( x \).

(ii) For each \( v \in V(X) \) let \( v^* \) to be the unique vertex of \( T \) such that \( G(v, v^*) \neq \varnothing \). It is clear that \( v \) and \( v^* \) are in the same \( G \)-vertex orbit and if \( v \in V(T) \) then \( v^* = v \). In general, if \( u \) and \( v \) are two vertices of \( X \) then \( (u^*)^* = u^* \) and if \( G(u, v) \neq \varnothing \) then \( u^* = v^* \).

(iii) For each edge \( y \) of \( Y \) such that \( o(y) \in V(T) \) define \([y] \) to be an element of 
\[ G(t(y), (t(y))^*), \text{that is, } [y](t(y))^* = t(y) \] to be chosen as follows

1. \([y] = 1 \) if \( y \in E(T) \),
2. \([y](y) = \tilde{y} \) if \( G(y, \tilde{y}) \neq \varnothing \).

We define \([y] = [\tilde{y}]^{-1} \) if \( G(y, \tilde{y}) = \varnothing \), and \([y] = [\tilde{y}] \) otherwise.

It is clear that \([y][\tilde{y}] = 1 \) if \( G(y, \tilde{y}) = \varnothing \) and \([y][\tilde{y}] = [y]^2 \in G_y \) otherwise.

(iv) For each edge \( y \) of \( Y \) let \(-y = [y]^{-1}(y) \) if \( o(y) \in V(T) \), otherwise \(-y = y \), and let \(+y = [y](y) \). So \(+y = y \) if \( o(y) \in V(T) \), otherwise \(+y = [y](y) \). It is clear that if \( o(y) \in V(T) \) and \( G(y, \tilde{y}) \neq \varnothing \) then \((o(y))^* = (t(y))^* = o(y) \) and \(-y = +y = \tilde{y} \).

If \( y \in E(T) \) then \(-y = +y = y \).

From above we see that \( G_{-y} \cap G_{+y} = G_y \cap [y]^{-1}G_y[y] \) if \( o(y) \in V(T) \) and
\[ G_{-y} \cap G_{+y} = G_y \cap [y]G_y[y]^{-1} \] otherwise.

(v) For each edge \( y \) of \( Y \) define \( \phi_y : G_{-y} \to G_{+y} \) by \( \phi_y(g) = [y]g[y]^{-1} \) and
\( \overline{G}_y = \{ g \in G_{-y} : \phi_y(g) = g \} \). It is clear that \( \phi_y \) is an isomorphism and
\( \overline{G}_y = \{ g \in G_{-y} : [y]g = g[y]\} \leq G_y \). Moreover, if \( y \in E(T) \) or \( G(\overline{y}, y) \neq \emptyset \) then \( \overline{G}_y = G_y \).

**Definition 1.1.** By a word \( w \) of \( G \) we mean an expression of the form
\[
w = g_0 \cdot y_1 \cdot g_1 \cdot \ldots \cdot y_n \cdot g_n, n \geq 0, y_i \in E(Y), \text{ for } i = 1, 2, \ldots, n \text{ such that}
\]
1. \( g_0 \in G_{(\alpha(y_1))^*} \),
2. \( g_i \in G_{(\alpha(y_i))^*} \), for \( i = 1, 2, \ldots, n \)
3. \( (t(y_1))^*(\alpha(y_{i+1}))^* \), for \( i = 1, 2, \ldots, n-1 \).

Regarding \( w \) as above we have the following.
(i) We define \( n \) to be the length of \( w \). We write \(|w| = n\). If \(|w| = 0\), i.e., \( w = g_0 \) then \( w \) is called a trivial word of \( G \).
(ii) We define the value of \( w \) denoted \([w]\) to be the element
\[
[w] = g_0[y_1]g_1 \cdots [y_n]g_n \text{ of } G.
\]
(iii) We define \( o(w) = (\alpha(y_1))^* \) and \( t(w) = (t(y_n))^* \). If \( o(w) = t(w) \), then \( w \) is called a closed word of \( G \) of type \( o(w) \).
(iv) The inverse of \( w \) denoted \( w^{-1} \) is defined to be the expression
\[
w^{-1} = g_n^{-1} \cdot \overline{y}_n \cdot g_{n-1}^{-1} \cdots g_1^{-1} \cdot \overline{y}_1 \cdot g_0^{-1} \text{ of } G.
\]

It is clear that \( w^{-1} \) is a word of \( G \). Moreover, \([w^{-1}] = [w]^{-1}\) and if \( w \) is closed then \( w^{-1} \) is closed and \((w^{-1})^{-1} = w\).
(v) If \( w_i = h_0 \cdot x_1 \cdot h_1 \cdot \ldots \cdot x_n \cdot h_m \) is a word of \( G \) such that \( t(w) = o(w_i) \) then \( w \cdot w_i \) is defined to be the word \( w \cdot w_i = g_0 \cdot y_1 \cdot g_1 \cdot \ldots \cdot y_n \cdot g_n \cdot h_0 \cdot x_1 \cdot h_1 \cdot \ldots \cdot x_n \cdot h_m \).

**Definition 1.2.** The word \( w = g_0 \cdot y_1 \cdot g_1 \cdot \ldots \cdot y_n \cdot g_n \) is called reduced word of \( G \) if \( w \) contains no expression of the form
1. \( y_i \cdot g_i \cdot \overline{y}_i \) if \( g_i \in G_{-y_i} \), for \( i = 1, \ldots, n \),
or
(2) \( y_i \cdot g_i \cdot y_i ^{-1} \) if \( g_i \in G_{y_i} \) with \( G ( y_i \cdot y_i ^{-1} ) \neq \emptyset \), for \( i = 1, \ldots, n \).

We take the trivial word to be reduced.

**Proposition 1.3.** Every element of \( G \) is the value of a closed and reduced word of \( G \). Moreover, if \( w \) is a non-trivial reduced and closed word of \( G \) then \([w]\) (the value of \( w \)) is not the identity element of \( G \).

**Proof.** See Mahmud (1991)

### 2. Main Results

The following lemma is essential for the proof of the main result of this paper.

**Lemma 2.1.** Let \( G \) be a group acting on a tree \( X, v \in V(X), f \in G \) and \( g \in G_v \) such that \( fg = gf \) and \( g \notin G_x \), for all \( x \in E(X), t(x) = v \). Then \( f \in G_v \).

**Proof.** Let \( T \) be a tree of representatives, and \( Y \) be a fundamental domain for the action of \( G \) on \( X \) such that \( T \subseteq Y \). Then there exists a unique vertex \( v^* \) of \( T \) such that \( G(v, v^*) \neq \emptyset \), i.e., \( v \) and \( v^* \) are in the same \( G \)-vertex orbit. Then there exists \( a \in G \) and \( b \in G_x \) such that \( g = aba^{-1} \). This implies that \( v = a(v^*) \) and \( a^{-1} fab = ba^{-1} fa \). Let \( h = a^{-1} fa \). Since \( g \notin G_x \), for all \( x \in E(X), t(x) = v \), therefore \( h \notin G_x \), for all \( y \in E(Y) \) such that \( t(y)) = v \). By Proposition 1.3, there exists a reduced word \( w = g_0 \cdot y_1 \cdot g_1 \cdot \ldots \cdot y_n \cdot g_n \) of \( G \) such that \( w \) is of type \( v^* \) and value \( h \), i.e., \((o(y)) = (t(y)) = v^* \) and \([w] = h \). Since \( h \notin G_{y_n} \), therefore \( w \cdot h \) and \( h \cdot w \) are reduced words of \( G \) of value \( I \), the identity element of \( G \). Therefore by Proposition 1.3 the word

\[
whw^{-1}h^{-1} = g_0 \cdot y_1 \cdot g_1 \cdot \ldots \cdot y_n \cdot g_n \cdot h \cdot g_n^{-1} \cdot y_n \cdot g_{n-1} \cdots g_1^{-1} \cdot y_1 \cdot g_0^{-1} \cdot h^{-1}
\]

is not reduced. The only way that the indicated word can fail to be reduced is that \( g_n h g_n^{-1} \in G_{y_n} \). Making successive \( y \)-reduction for \( y \in \{ y_1, \ldots, y_n \} \) we see that each \( L_i = g_{n-i} \cdot \phi_{y_{i+1}} (L_{i-1}) g_{n-i}^{-1} \) is in \( G_{y_{i+1}} \), for \( i = 1, \ldots, n \) with convention that \( L_0 = g_n h g_n^{-1} \) and \( L_n = g_0 \phi_{y_1} (L_{n-1}) g_0^{-1} \). Then \( h = g_0 \phi_{y_1} (L_{n-1}) g_0^{-1} \).

Since \((o(y_1)) = v^*, \phi_{y_1} (L_{n-1}) \in G_{y_1} \leq G_v \) and \( g_0 \in G_{y_1} \), therefore \( h \in G_{y_1} \).

This implies that \( a^{-1} fa \in G_{v^*} \). Therefore \( f \in aG_{v^*} \cdot a^{-1} = G_v \). This completes the proof.
\textbf{Theorem 2.2.} Let $G$ be a group acting on a tree $X$, $T$ be a tree of representatives and $Y$ be a fundamental domain for the action of $G$ on $X$ such that $T \subseteq Y$ and $G_x \neq G_{t(x)}$ for all $x \in E(X)$. Then the center $C(G)$ of $G$ is given by

$$C(G) = \left( \bigcap_{v \in V(T)} C(G_v) \right) \cap \left( \bigcap_{y \in E(Y)} \overline{G_y} \right)$$

where $C(G_v)$ is the center of $G_v$.

\textbf{Proof.} Let $f \in \left( \bigcap_{v \in V(T)} C(G_v) \right) \cap \left( \bigcap_{y \in E(Y)} \overline{G_y} \right)$ and $g \in G$. We need to show that $gf = fg$. By Proposition 1.3, $g$ can be written as $g = g_0 [y_1] g_1 \cdots [y_n] g_n$, where $g_0 \cdot y_1 \cdot g_1 \cdots y_n \cdot g_n$ is a reduced word of $G$. Then

$$gf = g_0 [y_1] g_1 \cdots [y_n] g_n f$$

$$= g_0 [y_1] g_1 \cdots [y_n] f g_n, \text{ because } g_n \in G_{(t(y_n))}^*, \text{ and } f \in C(G_{(t(y_n))}^*)$$

$$= g_0 [y_1] g_1 \cdots f [y_n] g_n, \text{ because } f \in \overline{G_{y_n}}$$

$$= g_0 [y_1] f g_1 \cdots [y_n] g_n, \text{ because } g_1 \in G_{(t(y_1))}^*, \text{ and } f \in C(G_{(t(y_1))}^*)$$

$$= g_0 [y_1] f g_1 \cdots [y_n] g_n, \text{ because } f \in \overline{G_{y_1}}$$

$$= f g_0 [y_1] g_1 \cdots [y_n] g_n, \text{ because } g_0 \in G_{(t(y_1))}^*, \text{ and } f \in C(G_{(t(y_1))}^*)$$

$$= fg.$$

Now let $f \in C(G)$, $v \in V(T)$ and $y \in E(Y)$. We need to show that $f \in C(G_v)$ and $f \in \overline{G_y}$. By assumption there exist $g \in G_v$ such that $g \not\in G_x$ for some $x \in E(X)$, $t(x) = v$.

Then $fg = gf$. Therefore by Lemma 2.1, $f \in C(G_v)$. This implies that $f \in G_{(t(y))}^*$.
Since we have \([y]\overline{f} = f[y]\), therefore the word \(w = 1 \cdot y \cdot f \cdot \overline{y} \cdot f^{-1}\) has value 1. Then Proposition 1.3 implies that \(f \in G_{y, y}\). Hence \(f \in \overline{G}_y\). Then \(G\) has the required center. This completes the proof.

We have the following two corollaries of Theorem 2.2.

**Corollary 2.3.** If \(G = \prod_{i \in I} (G_i, A_{i,j} = A_{k,j})\) is a tree product of the groups \(G_i, i \in I\), such that \(A_{i,j} \neq G_j\) and \(A_{i,j} \neq G_k\) for all \(j, k \in I\) then the center \(C(G)\) of \(G\) is given by

\[
C(G) = \left( \bigcap_{i \in I} C(G_i) \right) \cap \left( \bigcap_{j,k \in I} A_{j,k} \right)
\]

where \(C(G_i)\) is the center of \(G_i\).

**Proof.** There exists a tree \(X\) on which \(G\) acts, a tree of representatives \(T\) and a fundamental domain \(Y\) for the action of \(G\) on \(X\) such that \(T = Y\) and if \(v \in V(T)\) and \(y \in E(T)\) then \(G_v = G_t\) for some \(i \in I\), \(G_y = A_{j,k}\) for some \(j, k \in I\), \([y] = 1\) and \(\overline{G}_y = A_{j,k}\). Therefore by Theorem 2.2 \(G\) has the required center. This completes the proof.

**Corollary 2.4.** If \(G = (H, t_i | \text{rel } H, t_i A_i t_i^{-1} = B_i, i \in I)\) is an HNN group of base \(H\) and associated pairs \((A_i, B_i)\) of subgroups of \(H\) such that \(A_i \neq H\) for all \(i \in I\) then the center \(C(G)\) of \(G\) is given by

\[
C(G) = C(H) \cap \left( \bigcap_{i \in I} A_i \right)
\]

where \(C(H)\) is the center of \(H\) and \(\overline{A}_i = \{a \in A_i | t_i a t_i^{-1} = a\}\).

**Proof.** There exists a tree \(X\) on which \(G\) acts, a tree of representatives \(T\) consisting of exactly one vertex \(v\) and a fundamental domain \(Y\) for the action of \(G\) on \(X\) such that \(v \in V(Y)\), the unordered edges of \(Y\) are in one-to-one correspondence with \(I\), \(G_v = H\) and if \(y \in Y\), \(o(y) = v\) then \(G_y = A_i\), \([y] = t_i\), and \(\overline{G}_y = \overline{A}_i = \{a \in A_i | t_i a t_i^{-1} = a\}\), for some \(i \in I\). Therefore by Theorem 2.2 \(G\) has the required center. This completes the proof.
References

Magnus, W. Karrass, A. and Solitar, D. *Combinatorial group theory: Presentation of groups in terms of generators and relations.* Interscience (1966)


مرتكزات الزمر المؤثرة في الشجر

رشيد محمود

خلاص

في هذا البحث نثبت أنه إذا أثرت الزمرة على الشجرة، فإن زمرة معينة وان T وان X حيث أن X يشير إلى G مجال أساسي لتأثير G على Y وان Gx لجميع المراكز للزمرة

هذا يساوي

\[ C(G) = \left( \bigcap_{v \in V(T)} C(G_v) \right) \cap \left( \bigcap_{y \in E(Y)} \overline{G_y} \right) \]

حيث أن Q هو مركز الزمرة G Y وان Gx هو مركز زمرة جزئية معينة من