\textbf{\(\theta\)-Closure and Hausdorff Spaces}

Mohammad Saleh*

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\textbf{Abstract}

The purpose of this paper is to further the study of the concept of \(\theta\)-closure operators. We prove that a compact (closure compact) subset of a Hausdorff space is \(\theta\)-closed (closed). Moreover, we prove that a quasi-\(\theta\)-closed space \(X\) is Urysohn if every closure compact subset of \(X\) is \(\theta\)-closed. Also, we investigate some relations between some forms of continuity and their graphs. Among other results, it is shown that the graph of a strongly continuous function \(f\) is \(\theta\)-closed if the range space \(Y\) is Hausdorff.

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\textbf{Introduction}

The concepts of \(\delta\)-closure, \(\theta\)-closure, \(\delta\)-interior and \(\theta\)-interior operators were first introduced by Velichko. These operators have since been studied intensively by many authors. Although \(\theta\)-interior and \(\theta\)-closure operators are not idempotents, the collection of all \(\delta\)-open sets in a topological space \((X,\Gamma)\) forms a topology \(\Gamma_\delta\) on \(X\), called the semiregularization topology of \(\Gamma\), weaker than \(\Gamma\) and the class of all regular open sets in \(\Gamma\) forms an open basis for \(\Gamma_\delta\). Similarly, the collection of all \(\theta\)-open sets in a topological space \((X,\Gamma)\) forms a topology \(\Gamma_\theta\) on \(X\), weaker than \(\Gamma_\delta\). So far, numerous applications of such operators have been found in studying different types of continuous like maps, separation of axioms, and above all, to many important types of compact like properties.

Among other generalizations of continuity, the concepts of weak and closure continuity have been studied by many mathematicians like D.R. Andrew, J. Chew, L. Herrington, N. Levine, P. Long, T. Noire, M. Saleh, J. Tong, E.K. Whittlesey and

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* Mathematics Department, Birzeit University, P.O.Box 14, Birzeit, West Bank, Palestine.
others. In 1961, [4] introduced the concept of weak continuity as a generalization of continuity, later in 1966, Husain introduced almost continuity as another generalization, and Andrew and Whittlesey [2], the concept of closure continuity which is stronger than weak continuity. In 1968, Singal and Singal introduced a new almost continuity which is different from that of Husain. A few years later, P. E. Long and Carnahan [5] studied similarities and dissimilarities between the two concepts of almost continuity.

The purpose of this paper is to further the study of the concepts of closure and strong continuity. In Section 2, we study some basic properties of these maps. Our main results are contained in Sections 3, 4. Among other results we prove that a quasi-H-closed space 𝑿 is Urysohn iff every closure compact subset of 𝑿 is 𝜃-closed. Also, the graph mapping of a strongly continuous function 𝑓 is 𝜃-closed with respect to 𝑿× 𝑌 if the range space 𝑌 is Hausdorff.

For a set 𝐴 in a space 𝑿, let us denote by \( \text{Int}(𝐴) \) and \( \text{cls}(𝐴) \) for the interior and the closure of 𝐴 in 𝑿, respectively. Following Velichko, a point \( 𝑥 \) of a space 𝑿 is called a \( 𝜃 \)-adherent point of a subset 𝐴 of 𝑿 iff \( \text{cls}(𝐴) ∩ 𝑥 ≠ 𝜙 \), for every open set 𝑈 containing 𝑥. The set of all \( 𝜃 \)-adherent points of 𝐴 is called the \( 𝜃 \)-closure of 𝐴, denoted by \( \text{cls}_𝜃 𝐴 \). A subset 𝐴 of a space 𝑿 is called \( 𝜃 \)-closed iff \( 𝐴 = \text{cls}_𝜃 𝐴 \). The complement of a \( 𝜃 \)-closed set is called \( 𝜃 \)-open. Similarly, the \( 𝜃 \)-interior of a set 𝐴 in 𝑿, written \( \text{Int}_𝜃 𝐴 \), consists of those points \( 𝑥 \) of 𝐴 such that for some open set \( 𝑈 \) containing \( 𝑥 \), \( \text{cls}(𝐴) ⊆ 𝐴 \). A set 𝐴 is \( 𝜃 \)-open if \( 𝐴 = \text{Int}_𝜃 𝐴 \), or equivalently, \( 𝑋 \) is \( 𝜃 \)-closed. Clearly every \( 𝜃 \)-closed (\( 𝜃 \)-open) is closed (open).

A function \( 𝑓: 𝑿 → 𝑌 \) is weakly continuous at \( 𝑥 ∈ 𝑿 \) if given any open set \( 𝑉 \) in 𝑌 containing \( 𝑓(𝑥) \), there exists an open set \( 𝑈 \) in 𝑿 containing \( 𝑥 \) such that \( 𝑓(𝑈) ⊆ \text{cls}(𝑉) \). If this condition is satisfied at each \( 𝑥 ∈ 𝑿 \), then \( 𝑓 \) is said to be weakly continuous. A function \( 𝑓: 𝑿 → 𝑌 \) is closure continuous (\( 𝜃 \)-continuous) at \( 𝑥 ∈ 𝑿 \) if given any open set \( 𝑉 ⊆ 𝑌 \) containing \( 𝑓(𝑥) \), there exists an open set \( 𝑈 ⊆ 𝑿 \) containing \( 𝑥 \) such that \( 𝑓(\text{cls}(𝑈)) ⊆ \text{cls}(𝑉) \). If this condition is satisfied at each \( 𝑥 ∈ 𝑿 \), then \( 𝑓 \) is said to be closure continuous (\( 𝜃 \)-continuous). A function \( 𝑓: 𝑿 → 𝑌 \) is strongly continuous (strongly \( 𝜃 \)-continuous) at \( 𝑥 ∈ 𝑿 \) if given any open set \( 𝑉 ⊆ 𝑌 \) containing \( 𝑓(𝑥) \), there exists an open set \( 𝑈 ⊆ 𝑿 \) containing \( 𝑥 \) such that \( 𝑓(\text{cls}(𝑈)) ⊆ 𝑉 \). If this condition is satisfied at each \( 𝑥 ∈ 𝑿 \), then \( 𝑓 \) is said to be strongly continuous (strongly \( 𝜃 \)-continuous). A space 𝑿 is called Urysohn if for every \( 𝑥 ≠ 𝑦 ∈ 𝑿 \), there exist an open set \( 𝑈 \) containing \( 𝑥 \) and an open set \( 𝑉 \) containing \( 𝑦 \) such that \( \text{cls}(𝑈) ∩ \text{cls}(𝑉) = 𝜙 \). One
of the most weaker forms of compactness is closure compactness (quasi-H-closed or simply, QHC). A subset \( A \) of a space \( X \) is called a closure compact subset if every open cover has a finite subcollection whose closures cover \( A \). A closure compact Hausdorff space is called H-closed, first defined by Alexandroff and Urysohn. We get similar results to some of those in [3], [4], [5], [6], [10] applied to closure and strong continuities.

**Basic Results**

In this section, for the convenience of the reader, we give some basic results of the concepts of weak, closure and strong continuity from [3], [4], [9] that will be needed in this paper.

It is well-known that if \( f: X \rightarrow Y \) is continuous then \( f: X \rightarrow f(X) \) is continuous. This is not the case in closure(weak) continuity even over a Urysohn space as it is shown in the next example but it is true for strong continuity. Also, it is well-known that if \( f: X \rightarrow Y \) is continuous and \( A \subseteq X \) then \( f: A \rightarrow Y \) is continuous, this is still the case in closure, weak, and strong continuity.

**Lemma 2.1.** Let \( f: X \rightarrow Y \) be a closure (resp., weakly, strongly) continuous and let \( A \subseteq X \), then \( f: X \rightarrow Y \) is closure (resp., weakly, strongly) continuous.

**Example 2.2** [3, Example 2]. Let \( P \) be the upper half of the plane and let \( L \) be the \( X \)-axis. Let \( X = P \cup L \). If \( T \) is the half disc topology on \( X \) and let \( \Gamma \) be the relative topology that \( X \) inherits by virtue of being a subspace of \( \mathbb{R}^2 \). The identity function \( f: (X, \Gamma) \rightarrow (X, T) \) is closure(weak) continuous and thus by Lemma 2.1, \( f: (L, \Gamma) \rightarrow (X, T) \) is closure(weak)continuous, but \( f: (L, \Gamma) \rightarrow (L, T) \) is not weak(closure) continuous.

Clearly every strongly continuous function is continuous, but not conversely as the next example shows.

**Example 2.3.** Let \( X = \mathbb{R} \), where \( \Gamma \) is the topology with a basis whose members are of the form \((a, b)\) and \((a, b) - K, \ K = \{ -n : n \in \mathbb{Z}^+ \} \). Then \((R, \Gamma)\) is \( T_2 \) but not regular.

Let \( f: (R, \Gamma) \rightarrow (R, \Gamma), f(x) = x \). Then \( f \) is continuous but not strongly continuous.
Lemma 2.4. Let $X$ be a regular space and let $f: X \to Y$. Then $f$ is continuous iff $f$ is strongly continuous.

Lemma 2.5. Let $Y$ be a regular space and let $f: X \to Y$. Then $f$ is closure continuous iff $f$ is strongly continuous.

Lemma 2.6. Let $X$ be a regular space and let $f: X \to Y$. Then $f$ is closure continuous iff $f$ is weakly continuous.

Lemma 2.7. Let $X$ or $Y$ be a regular space and let $f: X \to Y$. Then $f$ is continuous iff $f$ is strongly continuous.

In [3] it is shown that if a function is continuous on $X$ then it is closure continuous on $X$. The next example shows that a continuous function at a point $x$ need not be closure continuous at $x$.

Example 2.8. Let $X = \{x, y, z\}$, $T_X = \{\emptyset, \{x\}, \{y\}, \{x, y\}, X\}$. Define $f: X \to Y$ by $f(x) = x$, $f(y) = f(z) = y$. Then $f$ is continuous at $x$ but not closure continuous at $x$, since $\text{cls}\{x\} = \{x, z\}$ but $\text{cls}\{x, y\} = X$ and $f(\\{x, z\}) = f(\\{x, y\}) = \{x, z\}$.

Lemma 2.9. Let $Y$ be a regular space and let $f: X \to Y$. Then $f$ is continuous iff $f$ is weakly continuous.

In [3] it is shown that weak continuity does not imply closure continuity. Therefore, strong continuity $\Rightarrow$ continuity $\Rightarrow$ closure continuity $\Rightarrow$ weak continuity, but not conversely.

It is well-known that the composition of continuous maps is continuous. Similar results hold for closure and strong continuity but it is not true for weak continuity.

Lemma 2.10. Let $f: X \to Y$ be weakly continuous and let $g: Y \to Z$ be closure continuous. Then $gof: X \to Z$ is weakly continuous.

Lemma 2.11. Let $f: X \to Y$ be continuous and let $g: Y \to Z$ be weakly continuous. Then $gof: X \to Z$ is weakly continuous.

Lemma 2.12. Let $f: X \to Y$ be closure continuous and let $g: Y \to Z$ be closure continuous. Then $gof: X \to Z$ is closure continuous.
Lemma 2.13. Let \( f:X \to Y \) be closure continuous and let \( g:Y \to Z \) be strongly continuous. Then \( gof: X \to Z \) is strongly continuous.

Lemma 2.14. Let \( f:X \to Y \) be weakly continuous and let \( g:Y \to Z \) be strongly continuous. Then \( gof \) is continuous.

The next example shows that the continuity of \( f \) in Lemma 2.8 cannot be weakened into closure continuity, and it also shows that the composition of weakly continuous need not be weakly continuous.

Example 2.15. Let \( X=\{x, y, z, w\} \) with topology \( \{\emptyset, \{x, y\}, \{z\}, \{z, w\}, X\} \) and let \( Y=\{a, b, c, d\} \) with topology \( \{\emptyset, \{a, b\}, \{b\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}, Y\} \). Define \( g:X \to Y \) by \( g(x)=a, g(y)=b, g(z)=c, g(w)=d \). Then \( g \) is weakly continuous but not closure continuous. Define \( f:(\mathbb{R}, \mathbb{U}) \to X \), where \( \mathbb{U} \) is the usual topology on \( \mathbb{R} \) by \( f(\text{rationals})=y, f(\text{irrationals})=w \). Then \( f \) is closure continuous but not continuous, and \( gof \) is not weakly continuous.

Main Results

It is well-known that a compact subset of a Hausdorff space is closed but not conversely, the next results are in that direction. Also, it is easy to see that a space \( X \) is a Hausdorff space iff for every \( x \in X \), \( \{x\} \) is \( \emptyset \)-closed.

Lemma 3.1. A space \( X \) is a Hausdorff space iff \( \{x\} \) is \( \emptyset \)-closed for every \( x \in X \).

Proof. The proof is straightforward.

Theorem 3.2. A space \( X \) is a Hausdorff space iff every compact subset of \( X \) is \( \emptyset \)-closed.

Proof. Let \( A \) be a compact subset of a Hausdorff space \( X \). We will show that \( X \setminus A \) is \( \emptyset \)-open. Let \( x \in X \setminus A \) then for each \( a \in A \) there exists \( U_{x,a} \) and \( V_a \) such that \( \text{cls}(U_{x,a}) \cap V_a = \emptyset \). The collection \( \{V_a : a \in A\} \) is an open cover of \( A \). Therefore, there exists a finite subcollection \( V_{i,1}, \ldots, V_{i,n} \), that cover \( A \). Let \( U=U_{i,1} \cap \ldots \cap U_{i,n} \), then \( \text{cls}(U) \cap A = \emptyset \). Thus \( X \setminus A \) is \( \emptyset \)-open, proving that \( A \) is \( \emptyset \)-closed. The converse follows from Lemma 3.1.
The proof of the next theorem is similar to Theorem 3.2 and thus will be omitted.

**Theorem 3.3.** A closure compact subset of a Hausdorff space is closed.

The next example shows that Hausdorffness cannot be weakened into $T_1$-space in the above results.

**Example 3.4.** Let $\mathbb{R}$ be the reals with the cofinite topology. Then every subset of $\mathbb{R}$ is compact and hence closure compact but the only closed subsets of $\mathbb{R}$ are the finite ones.

In [11] it is shown that over a completely Hausdorff $H$-closed space the class of $H$-closed sets coincides with the class of $\Theta$-closed sets. The next two results are a generalization of [11, Theorem 4].

**Theorem 3.5.** A quasi-$H$-closed space $X$ is Urysohn iff every closure compact subset is $\Theta$-closed.

**Proof.** Let $A$ be a closure compact subset of a Urysohn space $X$. We will show that $X \mathcal{A}$ is $\Theta$-open. Let $x \in X \setminus A$ then for each $a \in A$ there exist $U_a$ and $V_a$ such that $\text{cls}(U_a) \cap \text{cls}(V_a) = \emptyset$. The collection $\{V_a: a \in A\}$ is an open cover of $A$. Therefore, there exists a finite subcollection $V_{i_1}, ..., V_{i_n}$ whose closures cover $A$. Let $U = U_{i_1} \cap ... \cap U_{i_n}$. Since $\text{cls}(U) \cap U_{i_1} \cap ... \cap U_{i_n} = \emptyset$. Thus $X \setminus A$ is $\Theta$-open. Conversely, since point sets are compact it follows that $X$ is Hausdorff. Let $x \neq y \in X$. Then there exists an open set $U$ containing $x$ such that $y \not\in \text{cls}(U)$. By [7, 4.8(e)] it follows that $\text{cls}(U)$ is closure compact and thus $\Theta$-closed. Thus there exists an open set $V$ containing $y$ such that $\text{cls}(U) \cap \text{cls}(V) = \emptyset$, proving that $X$ is Urysohn.

**Theorem 3.6.** A $\Theta$-closed subset of closure compact is closure compact.

**Proof.** Let $A$ be a $\Theta$-closed subset of $X$ and let $\mathcal{C}$ be an open cover of $A$. Then $\mathcal{C}$ plus $X \mathcal{A}$ is an open cover of $X$. Since $X$ is closure compact, there exists a finite subcollection whose closures cover $X$. If $X \mathcal{A}$ is one of this collection, take it out and the remaining is a closure cover of $A$, otherwise this cover is also a closure cover of $A$, proving that $A$ is closure compact.
Corollary 3.7. Every clopen subset of a closure compact space is closure compact.

Lemma 3.8 [9, Theorem 1]. Let $f : X \to Y$. Then the following are equivalent:

a) $f(\text{cls}_0 A) \subseteq \text{cl} f(A)$, for every $A \subseteq X$,

b) The inverse image of closed is $\theta$-closed,

c) The inverse image of open is $\theta$-open,

d) $f$ is strongly continuous.

Corollary 3.9. Let $f : X \to Y$ be strongly continuous where $Y$ is a $T_1$-space. Then $f$ has $\theta$-closed point inverse.

The hypothesis in Corollary 3.9 can't be weakened into $T_0$ as it is shown in the next example.

Example 3.10. Let $X$ be the reals with the lower limit topology and $Y = \mathbb{R}$ with the right ray topology. Define $f : X \to Y$ as follows $f(x) = 0$ for all $x < 0$, $f(x) = 1$, for all $x > 0$. Then $f$ is strongly continuous, and $\{1\}$ is compact but $f^{-1}(0) = (-\infty, 0)$ is not even closed.

Corollary 3.11. Let $f : X \to Y$ be continuous where $X$ or $Y$ is regular. Then the inverse image of every closed subset of $Y$ is $\theta$-closed.

The hypothesis in Corollary 3.11 can't be weakened into Urysohn as it is shown in Example 2.3, since $K$ is closed but not $\theta$-closed.

Corollary 3.12. Let $f : X \to Y$ be strongly continuous where $Y$ is a $T_2$-space. Then the inverse image of every closure compact subset of $Y$ is $\theta$-closed.

Proof. Let $K$ be a closure compact subset of $Y$. Theorem 3.3 implies that $K$ is closed and thus Lemma 3.8 implies that $f^{-1}(K)$ is $\theta$-closed.

Corollary 3.13. Let $f : X \to Y$ be continuous where $Y$ is a $T_2$-space and $X$ is regular. Then the inverse image of every closure compact subset of $Y$ is $\theta$-closed.

Proof. The proof follows from Corollary 3.12 since in this case $f$ will be strongly continuous.
Corollary 3.14. Let \( f : X \to Y \) be strongly continuous where \( X \) is a \( T_2 \)-space and \( X \) is closure compact. Then the inverse image of every closure compact subset of \( Y \) is closure compact.

Proof. Let \( K \) be a closure compact subset of \( Y \). Corollary 3.13 implies that \( f^{-1}(K) \) is \( \emptyset \)-closed. Once more, Theorem 3.6 implies that \( f^{-1}(K) \) is closure compact.

The hypothesis in Corollary 3.14 that \( X \) being closure compact can't be removed nor we could weaken the Hausdorffness of \( Y \) in Corollaries 3.11, 3.12, 3.13 as it is shown in the next examples.

Example 3.15. Let \( X = \mathbb{R} \) be the reals with the usual topology. Define \( f : X \to Y \) as follows \( f(x) = 0 \) for all \( x \leq 0 \), \( f(x) = x \), for all \( x > 0 \). Then \( f \) is strongly continuous, and \( \{0\} \) is compact but \( f^{-1}(\emptyset) = (\emptyset, 0] \) is not closure compact.

Example 3.16. Let \( X \) be the reals with the usual topology and \( Y \) be the reals with the cocomplete topology. Let \( f : X \to Y \) be the identity map. Then \( f \) is strongly continuous, and any subset of \( Y \) is closure compact but \( f^{-1}(0, \infty) = (0, \infty) \) is not closure compact.

Theorem 3.17. Let \( f : X \to Y \) be closure continuous. Then the following holds

(a) \( f(\text{cls}_B A) \subseteq \text{cls}_B f(A) \), for every \( A \subseteq X \),

(b) The inverse image of \( \emptyset \)-closed is \( \emptyset \)-closed,

(c) The inverse image of \( \emptyset \)-open is \( \emptyset \)-open.

Proof.

(a) Let \( f : X \to Y \) be closure continuous and let \( x \in \text{cls}_B A \). Let \( V \) be an open set containing \( f(x) \). By closure continuity of \( f \), there exists an open set containing \( x \) such that \( f(\text{cls}(U)) \subseteq \text{cls}(V) \). Therefore, \( \text{cls}(V) \) meets \( A \) and thus \( \text{cls}(V) \) meets \( f(A) \). Hence \( f(x) \in \text{cls}_B f(A) \), as we claim.

(b) Let \( B \) be a \( \emptyset \)-closed set and \( A = f^{-1}(B) \). Let \( x \in \text{cls}_B A \). By part (a), \( f(x) \in f(\text{cls}_B A) \subseteq f(\text{cls}_B f(A)) \subseteq \text{cls}_B f(B) = B \). Therefore, \( x \in f^{-1}(B) = A \). Thus \( \text{cls}_B A = A \).

(c) Follows directly from (b) since (b) \( \iff \) (c)
Corollary 3.18. Let \( f: X \to Y \) be closure continuous where \( Y \) is a \( T_2 \)-space. Then \( f \) has \( \theta \)-closed point inverses.

**Proof.** The proof follows directly from Lemma 3.1 and Theorem 3.17.

Corollary 3.19. Let \( f: X \to Y \) be closure continuous where \( Y \) is a Urysohn space. Then the inverse image of every closure compact subset of \( Y \) is \( \theta \)-closed.

**Proof.** Let \( K \) be a closure compact subset of \( Y \). Theorem 3.6 implies that \( K \) is \( \theta \)-closed and thus Theorem 3.17 implies that \( f^{-1}(K) \) is \( \theta \)-closed.

Corollary 3.20. Let \( f: X \to Y \) be closure continuous where \( Y \) is a Urysohn space and \( X \) is closure compact. Then the inverse image of every closure compact subset of \( Y \) is closure compact.

**Proof.** Let \( K \) be a closure compact subset of \( Y \). Corollary 3.19 implies that \( f^{-1}(K) \) is \( \theta \)-closed. Once more, Theorem 3.6 implies that \( f^{-1}(K) \) is closure compact.

Corollary 3.21. Let \( f: X \to Y \) be closure continuous where \( Y \) is a Hausdorff space. Then the inverse image of every compact subset of \( Y \) is \( \theta \)-closed.

**Proof.** Let \( K \) be a compact subset of \( Y \). Theorem 3.2 implies that \( K \) is \( \theta \)-closed and thus Theorem 3.17 implies that \( f^{-1}(K) \) is \( \theta \)-closed.

Corollary 3.22. Let \( f: X \to Y \) be weakly continuous where \( Y \) is a \( T_2 \)-space and \( X \) is regular. Then \( f \) has \( \theta \)-closed point inverses.

**Proof.** Since \( X \) is regular, \( f \) is closure compact. Thus, Corollary 3.18 leads that \( f \) has \( \theta \)-closed point inverses.

The hypothesis in the above Corollaries can't be weakened into \( T_1 \) as it is shown in the next example.

**Example 3.23.** Let \( X = \mathbb{R}_a \) be the reals with the usual topology and \( \mathbb{R}_c \) the reals with the cocompactable topology. Define \( f: \mathbb{R}_a \to \mathbb{R}_c \) as follows, \( f(\text{rational}) = 0, f(\text{irrational}) = 1 \). Then \( f \) is closure continuous and \( \{0\} \) is compact but \( f^{-1}(\{0\}) \) is neither closed nor closure compact.
Corollary 3.24. Let $f:X \to Y$ be closure continuous where $Y$ is a Hausdorff space and $X$ is closure compact. Then the inverse image of every compact subset of $Y$ is closure compact.

Proof. Let $K$ be a compact subset of $Y$. Corollary 3.21 implies that $f^{-1}(K)$ is $\theta$-closed. Once more, Theorem 3.6 implies that $f^{-1}(K)$ is closure compact.

Corollary 3.25. Let $f:X \to Y$ be continuous where $Y$ is a $T_2$-space. Then $f$ has $\theta$-closed point inverses.

Proof. The result follows directly from Theorem 3.17 since every continuous function is closure continuous.

The next example shows that $Y$ being Hausdorff can't be weakened into $T_1$ in Corollary 3.25.

Example 3.26. Let $X=Y=\mathbb{R}$ be the reals with the cofinite topology. Let $f: \mathbb{R} \to \mathbb{R}$ be the identity function. Then $f$ is continuous but $f^{-1}(\{0\})$ is not $\theta$-closed.

Theorem 3.27. Let $f:X \to Y$ be weakly continuous. Then the following holds

a) $f(\text{cls } A) \subseteq \text{cls}_0 f(A)$, for every $A \subseteq X$,

b) The inverse image of $\theta$-closed is closed,

c) The inverse image of $\theta$-open is open.

Proof.

(a) Let $f:X \to Y$ be weakly continuous and let $x \in \text{cls } A$. Let $V$ be an open set containing $f(x)$. By weak continuity of $f$ there exists an open set containing $x$ such that $f(U) \subseteq \text{cls} V$. Therefore, $U$ meets $A$ and thus $\text{cls}(V)$ meets $f(A)$. Hence $f(x) \in \text{cls}_0 f(A)$, as we claim.

(b) Let $B$ be a $\theta$-closed set and $A=f^{-1}(B)$. Let $x \in \text{cls}_0 A$. By part (a), $f(x) \in f(\text{cls}_0 A) \subseteq \text{cls}_0 f(A) \subseteq \text{cls}_0 B = B$. Therefore, $x \in f^{-1}(B) = A$. Thus $\text{cls}_0 A = A$.

(c) Follows directly from (b) since (b) $\iff$ (c)

Corollary 3.28 [3, Theorem 6]. Let $f:X \to Y$ be weakly continuous where $Y$ is a $T_2$-space. Then $f$ has closed point inverses.
Corollary 3.29. Let $f: X \to Y$ be weakly continuous where $Y$ is a Urysohn space. Then the inverse image of every closure compact subset of $Y$ is closed.

Proof. Let $K$ be a closure compact subset of $Y$. Theorem 3.5 implies that $K$ is $\theta$-closed and thus Theorem 3.27 implies that $f^{-1}(K)$ is closed.

Corollary 3.30. Let $f: X \to Y$ be weakly continuous where $Y$ is a Urysohn space and $X$ is compact. Then the inverse image of every closure compact subset of $Y$ is compact.

Proof. Let $K$ be a closure compact subset of $Y$. Corollary 3.29 implies that $f^{-1}(K)$ is closed. Thus $f^{-1}(K)$ is compact.

Corollary 3.31. Let $f: X \to Y$ be weakly continuous where $Y$ is a Hausdorff space. Then the inverse image of every compact subset of $Y$ is closed.

Proof. Let $K$ be a compact subset of $Y$. Theorem 3.2 implies that $K$ is $\theta$-closed and thus Theorem 3.27 implies that $f^{-1}(K)$ is closed.

Corollary 3.32. Let $f: X \to Y$ be weakly continuous where $Y$ is a $T_2$-space and $X$ is compact. Then the inverse image of every compact subset of $Y$ is compact.

Proof. Let $K$ be a compact subset of $Y$. Corollary 3.32 implies that $f^{-1}(K)$ is closed. Thus $f^{-1}(K)$ is compact.

Examples 3.23, 2.15 show that the hypothesis in the above Corollaries can't be weaken into $T_1$.

The converses of Theorems 3.17, 3.27 are not true in general as it is shown in the next example.

Example 3.26. Let $X=\mathbb{R}$ with the cocountable topology, $Y=\{0, 1, 2\}$ with $\Gamma=\{\emptyset, \{0\}, \{1\}, \{0, 1\}, Y\}$. Define $f: X \to Y$ as $f(\text{rationals})=0$, $f(\text{irrationals})=1$. Then $f$ satisfies condition (b) in Theorem 3.7 but not even weakly continuous.

Functions Versus Graphs.

In [3], [9] it is shown that a function is weakly (resp., closure) continuous if its graph mapping is weakly (resp., closure) continuous, but it is not the case for strong
continuity. In this section we study some relations between these functions and their graphs. First we introduce the following generalizations of closed graphs.

The graph $G(f)=\{(x, f(x)): x \in X\}$ of a function $f:X \to Y$ is called $\theta$-closed with respect to $X \times Y$ (resp. $X$, $Y$) if for each $(x, y) \notin G(f)$ there exist open sets $U$ and $V$ containing $x$ and $y$, respectively, such that $\text{cls}U \times \text{cls}V \cap G(f) = \emptyset$ (resp., $\text{cls}U \times V \cap G(f) = \emptyset$, $U \times \text{cls}V \cap G(f) = \emptyset$).

**Theorem 4.1.** If $f:X \to Y$ is strongly continuous and $Y$ is Hausdorff, then $G(f)$ is $\theta$-closed with respect to $X \times Y$.

**Proof.** Let $x \in X$ and $y \neq f(x)$. By the Hausdorffness of $Y$, there exist an open set $V$ containing $f(x)$ and an open set $W$ containing $y$ such that $V \cap \text{cls}W = \emptyset$. Since $f$ is strongly continuous there is an open set $U$ containing $x$ such that $f(\text{cls}U) \subseteq V$. Therefore, $f(\text{cls}U) \cap W = \emptyset$, proving that $G(f)$ is $\theta$-closed in $X \times Y$.

**Theorem 4.2.** If $f:X \to Y$ is closure continuous and $Y$ is Hausdorff, then $G(f)$ is $\theta$-closed with respect to $X$.

**Proof.** Let $x \in X$ and $y \neq f(x)$. By the Hausdorffness of $Y$, there exist an open set $V$ containing $f(x)$ and an open set $W$ containing $y$ such that $W \cap \text{cls}V = \emptyset$. Since $f$ is closure continuous there is an open set $U$ containing $x$ such that $f(\text{cls}U) \subseteq \text{cls}V$. Therefore, $f(\text{cls}U) \cap W = \emptyset$, proving that $G(f)$ is $\theta$-closed with respect to $X$.

If we assume $Y$ is Urysohn then we get a stronger result.

**Theorem 4.3.** If $f:X \to Y$ is closure continuous and $Y$ is Urysohn, then $G(f)$ is $\theta$-closed with respect to $X \times Y$.

**Theorem 4.4.** If $f:X \to Y$ is weakly continuous and $Y$ is Hausdorff, then $G(f)$ is a closed subset of $X \times Y$.

**Proof.** Let $x \in X$ and $y \neq f(x)$. By the Hausdorffness of $Y$, there exist an open set $V$ containing $f(x)$ and an open set $W$ containing $y$ such that $W \cap \text{cls}V = \emptyset$. Since $f$ is weakly continuous there is an open set $U$ containing $x$ such that $f(U) \subseteq \text{cls}V$. Therefore, $f(U) \cap W = \emptyset$, proving that $G(f)$ is a closed subset of $X \times Y$.

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Theorem 4.5. If \( f : X \to Y \) is weakly continuous and \( Y \) is Urysohn, then \( G(f) \) is \( \theta \)-closed with respect to \( Y \).

Proof. Let \( x \in X \) and \( y \neq f(x) \). Since \( Y \) is Urysohn, there exist an open set \( V \) containing \( f(x) \) and an open set \( W \) containing \( y \) such that \( \text{cls} V \cap \text{cls} W = \emptyset \). Since \( f \) is weakly continuous there is an open set \( U \) containing \( x \) such that \( f(U) \subseteq \text{cls} V \). Therefore, \( f(U) \cap \text{cls} W = \emptyset \), proving that \( G(f) \) is \( \theta \)-closed with respect to \( Y \).

\( \theta \)-Closed and \( \theta \)-Open Mappings

In this section we introduce some generalizations of open and closed maps.

Definition 5.1. A function \( f \) is said to be \( \theta \)-open if the image of every open set is \( \theta \)-open. Similarly, a function \( f \) is said to be \( \theta \)-closed if the image of every closed set is \( \theta \)-closed.

Definition 5.2. A function \( f \) is said to be w.open if the image of every \( \theta \)-open set is open. Similarly, a function \( f \) is said to be w.closed if the image of every \( \theta \)-closed set is closed.

The next theorems are a remarkable improvement of [10, Theorem 2.13].

Theorem 5.1. Let \( f : X \to Y \) be weakly continuous 1-1, onto. If \( X \) is compact, \( Y \) Urysohn then \( f \) is \( \theta \)-open.

Proof. Let \( U \) be an open subset of \( X \), and thus \( X \setminus U \) is a closed subset of \( X \). Hence, \( X \setminus U \) is compact. Since \( f \) is weakly continuous, \( f(X \setminus U) \) is closure compact. Therefore, Theorem 3.5 implies that \( f(X \setminus U) = Y \) \( f(U) \) is \( \theta \)-closed, and thus \( f(U) \) is \( \theta \)-open.

Following a similar proof as in Theorem 5.1 we get the following results.

Theorem 5.2. Let \( f : X \to Y \) be weakly continuous. If \( X \) is compact, \( Y \) Urysohn then \( f \) is \( \theta \)-closed.

Theorem 5.3. Let \( f : X \to Y \) be weakly continuous 1-1, onto. If \( X \) is compact, \( Y \) Hausdorff then \( f \) is open.

Theorem 5.4. Let \( f : X \to Y \) be weakly continuous. If \( X \) is compact, \( Y \) Hausdorff then \( f \) is closed.
Theorem 5.5. Let $f:X \to Y$ be a continuous $1-1$, onto. If $X$ is compact, $Y$ Hausdorff then $f$ is $\theta$-open.

Theorem 5.6. Let $f:X \to Y$ be continuous. If $X$ is compact, $Y$ Hausdorff then $f$ is $\theta$-closed.

Theorem 5.7. Let $f:X \to Y$ be closure continuous $1-1$, onto. If $X$ is closure compact, $Y$ Urysohn then the image of $\theta$-open is $\theta$-open.

Theorem 5.8. Let $f:X \to Y$ be closure continuous $1-1$, onto. If $X$ is closure compact, $Y$ Hausdorff then $f$ is w-open.

Theorem 5.9. Let $f:X \to Y$ be closure continuous. If $X$ is closure compact, $Y$ Hausdorff then $f$ is w-closed.

Theorem 5.10. Let $f:X \to Y$ be closure continuous. If $X$ is closure compact, $Y$ Urysohn then the image of $\theta$-closed is $\theta$-closed.

Theorem 5.11. Let $f:X \to Y$ be a strongly continuous $1-1$, onto. If $X$ is closure compact, $Y$ Hausdorff then the image of theta-open is theta-open.

Theorem 5.12. Let $f:X \to Y$ be strongly continuous. If $X$ is closure compact, $Y$ Hausdorff then the image of $\theta$-closed is $\theta$-closed.

Recall that a subset $U$ of a space $X$ is called regularly open if $\text{Int} (\text{cls}(U)) = U$, and is called regularly closed if $\text{cls} (\text{Int}(U)) = U$.

The next theorems are an improvement of [10, Theorem 3.1].

Theorem 5.13. Let $f:X \to Y$ be closure continuous $1-1$, onto. If $X$ is closure compact, $Y$ completely Hausdorff then the image of regularly open is $\theta$-open.

Theorem 5.14. Let $f:X \to Y$ be closure continuous. If $X$ is closure compact, $Y$ Urysohn then the image of regularly closed is $\theta$-closed.


References


