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WE HEREBY RECOMMEND THAT THE DISSERTATION PREPARED UNDER OUR SUPERVISION BY MASHHOOR A. REFAI ENTITLED GROUP ACTIONS ON FINITE CW-COMPLEXES BE ACCEPTED AS FULFILLING IN PART REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

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CHAPTER 1

CW-COMPLEXES AND GROUP ACTIONS

This chapter will be a review for many of the basic definitions. Also there will be given some examples and remarks, necessary for our study of group actions on finite CW-complexes. Other definitions will be made as necessary in the text of this thesis.

Definition 1.1: Let G be a finite group and X be a set. An action of G on X is a function \( \alpha: G \times X \rightarrow X \), such that

(i) \( \alpha(1, x) = x \) for every \( x \in X \), where \( 1 \) is the identity element of G.

(ii) \( \alpha(gh, x) = \alpha(g, \alpha(h, x)) \) for all \( g, h \in G \) and \( x \in X \).

Let us denote \( \alpha(g, x) \) by \( g \cdot x \). The action is said to be free if \( gx = x \) for any \( x \in X \), implies \( g = e \). Or equivalently the action is free if the isotropy subgroup at \( x \), \( G_x = \{ g \in G \mid gx = x \} = \{ e \} \), for every \( x \in X \). The action is called trivial if \( gx = x \), for all \( g \in G \) and for all \( x \in X \).

Remarks 1.2:

a) Recall that G with the discrete topology is a topological group. So if X is given to be a topological space, then \( \alpha \) in the previous definition should be continuous, and X is said to be a G-space.

b) If X is an abelian group, one also assumes \( g \cdot 0 = 0 \) and \( g(a+b) = ga + gb \), for all \( g \in G \) and \( a, b \in X \). In this case, "X is a G-module".
c) If $X$ is a vector space over a field $k$, it is also required (in addition to b) that $g(\beta x) = \beta g(x)$ for all $\beta \in k$, $g \in G$ and $x \in X$.

**Definition 1.3:** Let $G$ be a finite group of order $n$, and let $k$ be a field. Define the group algebra $k[G]$ to be a $k$-vector space with a basis consisting of the elements of $G$. An arbitrary element in $k[G]$ can be written as

$$\sum_{g \in G} \alpha_g g$$

where $\alpha_g \in K$, and the operations are defined by,

$$\left[ \sum_{g \in G} \alpha_g g \right] \cdot \left[ \sum_{h \in G} \beta_h h \right] = \sum_{g, h \in G} (\alpha_g \beta_h)(gh)$$

and

$$\left[ \sum_{g \in G} \alpha_g g \right] + \left[ \sum_{g \in G} \beta_g g \right] = \sum_{g \in G} (\alpha_g + \beta_g) g$$

A **chain complex** over $k[G]$ is a sequence of $k[G]$-modules $\{C_i\}_{i \in \mathbb{Z}}$ with $k[G]$-homomorphisms $\delta_i : C_i \rightarrow C_{i-1}$, such that $\delta_{i-1} \delta_i = 0$.

If $X$ is a vector space on which $G$ acts, then $X$ becomes a $k[G]$-module and vice versa.

If $X$ is a $G$-space, then for each $g \in G$, there is a homeomorphism $g : X \rightarrow X$ which takes $x$ into $gx$. This map then induces an isomorphism $g_* : H_i^G(X, k) \rightarrow H_i^G(X, k)$ for each $i$, and hence the map

$\eta : G \times H_i^G(X, k) \rightarrow H_i^G(X, k)$ which takes $(g, \alpha)$ into $g_\alpha(\alpha)$ gives a $k[G]$-module structure on $H_i^G(X, k)$ for all $i$.

**Example 1.4:** Consider $X = S^{2t-1} \subset \mathbb{C}^t$ as

$$S^{2t-1} = \left\{ (Z_1, \ldots, Z_t) : Z_i \in \mathbb{C}, \sum \|Z_i\|^2 = 1 \right\}.$$

Let $G = \mathbb{Z}/n \cong \left\{ 1, e^{\frac{2\pi i}{n}}, e^{\frac{4\pi i}{n}}, \ldots, e^{\frac{2\pi i(n-1)}{n}} \right\}$. 


Define an action of $G$ on $X$ by:

$$e^{\frac{2\pi ik}{n}} \cdot (Z_1, Z_2, \ldots, Z_t) = (e^{\frac{2\pi ik}{n}} \cdot Z_1, \ldots, e^{\frac{2\pi ik}{n}} \cdot Z_t).$$

Clearly $e^{\frac{2\pi i}{n}} \cdot (Z_1, \ldots, Z_t) = (Z_1, \ldots, Z_t)$ and

$$\left(e^{\frac{2\pi ik}{n}} \cdot e^{\frac{2\pi ik}{n_2}} \right) \cdot (Z_1, \ldots, Z_t) = e^{\frac{2\pi ik}{n}} \cdot \left(e^{\frac{2\pi ik}{n}} \cdot (Z_1, \ldots, Z_t)\right).$$

If $e^{\frac{2\pi ik}{n}} \cdot (Z_1, \ldots, Z_t) = (Z_1, \ldots, Z_t)$, then $e^{\frac{2\pi ik}{n}} = 1$, and hence the action is free.

Recall the following definition and lemma.

**Definition 1.5:** Let $n \geq 1$, and $f: S^n \to S^n$ be a continuous map. Let $\alpha$ be one of the generators of $H_n(S^n) \cong \mathbb{Z}$. Then $f_\#(\alpha) = d\alpha$ for some integer $d$. The integer $d$ is called the degree of the map $f$, and it is independent of the choice of the generator, because $f_\#(-\alpha) = d(-\alpha)$. We say that $f$ has a **fixed point**, if there exist a point $x \in S^n$, for which $f(x) = x$.

**Lemma 1.6:** If $f: S^n \to S^n$ has no fixed point, then degree $f = (-1)^{n+1}$.

**Proof:** See Munkres [25].

In the previous example, the map $g: X \to X$ which takes $x$ into $g_x$ induces an isomorphism $g_\#: H_\#(S^{2t-1}, \mathbb{Z}) \to H_\#(S^{2t-1}, \mathbb{Z})$. Now

$$H_\#(S^{2t-1}, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } * = 0, 2t-1 \\ 0 & \text{otherwise} \end{cases}$$
Since the action has no fixed points, the lemma above says

\[ \text{degree } g = (-1)^{(2t-1)t+1} = 1. \]

Thus \( g_x(\alpha) = d \alpha = 1 \cdot \alpha \), for a generator \( \alpha \in H_{2t-1}(S^{2t-1}, \mathbb{Z}) \). Therefore, \( g_x \) - Identity and hence the action of \( G \) is trivial on homology. At the end of this chapter you will see an example of an action, whose induced action on homology is not trivial.

Remark 1.7: The action of the previous example can be extended to an action of \( \mathbb{Z}/n \) on \( S^{2t-1} \) by

\[ (g_1, g_2, \ldots, g_\ell) \cdot (x_1, x_2, \ldots, x_\ell) = (g_1 \cdot x_1, g_2 \cdot x_2, \ldots, g_\ell \cdot x_\ell). \]

Given a CW-complex \( X \), one can construct a cellular chain complex as follows, see Munkres [25] for more details.

For each \( p \geq 0 \), let \( X^p \) be the union of all cells in \( X \) of dimension \( \leq p \), and \( X^n = \emptyset \) for \( n < 0 \).

Let \( C_p(X) = H_p(X^p, X^{p-1}) \), and define the boundary map

\[ \partial_p : C_p(X) \rightarrow C_{p-1}(X) \text{ to be the composite} \]

\[ H_p(X^p, X^{p-1}) \xrightarrow{\Delta} H_{p-1}(X^{p-1}) \xrightarrow{(i_{p-1})^\ast} H_{p-1}(X^{p-1}, X^{p-2}) \]

where \( i_{p-1} \) is the inclusion from \( (X^{p-1}, \emptyset) \) into \( (X^{p-1}, X^{p-2}) \) and \( \Delta_p \) is the connecting homomorphism in the following long exact sequences.
Since the vertical sequence is exact, $\Delta_{p-1} \circ (i_{p-1})^* = 0$, and hence,

$$\partial_{p-1} \circ \partial_p = \left[ (i_{p-2})^* \circ \Delta_{p-1} \right] \circ \left[ (i_{p-1})^* \circ \Delta_p \right]$$

$$= (i_{p-2})^* \circ \left[ \Delta_{p-1} \circ (i_{p-1})^* \right] \circ \Delta_p$$

$$= (i_{p-2})^* \circ 0 \circ \Delta_p = 0.$$ 

So we have a cellular chain complex

$$C_*(X) = \left\{ C_p(X), \partial_p \right\}.$$ 

Using this cellular chain complex, one can compute the homology and cohomology groups of $X$. 