Abstract. The aim of this paper is to give some results on the generalization of theorem the Guelfond.

1. Introduction and Results

In 1914 G.Polya show that the entire function $f$ of a complex variable, with $f(\mathbb{N}) \subset \mathbb{Z}$ and $\lim_{r \to \infty} \sup (\log |f|_r / r) \leq \log 2$ is necessarily a polynom.

F.Gramain [7] presente the situation in 1988 an attempt of the same kind: how to show by a method of transcendence, a result obtained in 1933 by A.O.Guelfond on entire functions taking integer values in all points of a geometric progression, like the previous multiplicative additive problem (see[6] theorem VIII). On the other hand, the same type of results for entire functions of several variables were obtained by P.Bundschuh1980 and J.P.Bézivin 1983 the first uses Newton interpolation series in several variables and the second linear r écurentes suites. J.P.Bézivin[1] in 1984 studied a multivariate generalization of a result the Guelfond. Tanguy Rivoal and

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281
Michael Welter [11] have found the following result: let $F$ be a holomorphic function on $\{ z \in \mathbb{C}^d : R(z_j) > 0, j = 1, ..., d \}$, suppose that the following conditions are satisfied,

$(i)$ $F(\mathbb{N}^d) \subset \mathbb{Z}$.

$(ii)$ There are real $c > 0, \alpha \geq 0$ and $\beta$ such that we have for all $z$ satisfying $R(z_j) > 0$:

$$|F(z)| \leq c \prod_{j=1}^{d} \frac{e^{\Psi(\theta_j)}}{R(z_j)^{\alpha}(1 + |z_j|)^{\beta}} \text{ or } \theta_j \in \left[ \frac{-\pi}{2}, \frac{\pi}{2} \right],$$

$$\Psi(\theta_j) = \cos(\theta_j) \log(2 \cos(\theta_j)) + \theta_j \sin(\theta_j),$$

so if $\frac{1}{2} - \beta < \alpha < \frac{1}{2}$, the function $F$ is a polynom with rational coefficients.

B.Djebbar [3] says That a entire harmonic function $h$ on $\mathbb{C}$ the exponential type $\tau < \pi$.

If $h(z) = 0$, for $z = 0, 1, 2, ...$, and $h(z) = h(\overline{z})$, then $h = 0$.

It was proven in [5] that if $U$ an entire separately harmonic functions on $\mathbb{R}^{2N}$ such that:

$$\exists A > 0 \text{ such that } M_\Delta(U, r) \leq A \exp(\sigma r), \forall r > 0$$

and if $\sigma < \log 2$, then $U$ is a polynom.

Where for any complete bounded domain $\Delta$ of center 0 in $\mathbb{C}^N$, $M_\Delta(f, r) = \sup_{z \in \Delta} |f(z)|$.

2. Definitions and properties

2.1. Order and type of an entire function. Let $\chi : [0, +\infty[ \rightarrow [0, +\infty[ \cup \{ +\infty \}$ be an increasing function. We define the growth order $\rho(\chi)$ of $\chi$ by the formula:

$$\rho(\chi) = \limsup_{r \to \infty} \frac{\log \chi(r)}{\log r},$$
if $\chi$ is of finite order, $0 < \varrho(\chi) < \infty$) the growth type $\sigma(\chi)$ is defined by:

$$\sigma(\chi) = \limsup_{r \to \infty} \frac{\chi(r)}{r^{\varrho}}.$$  

For any entire function $f$ on $\mathbb{C}^N$, the $N$-growth of $f$ is defined by the growth of the function $\chi(r) = \log^+ M_N(f, r)$ where $N$ is a norm on $\mathbb{C}^N$ and $M_N(f, r) = \sup_{N(z) \leq r} |f(z)|$. For any complete bounded domain $\Delta$ of center 0 in $\mathbb{C}^N$, the $\Delta$-growth of $f$ is defined by the growth of the function $\chi(r) = \log^+ M_\Delta(f, r)$ where $M_\Delta(f, r) = \sup_{z \in r, \Delta} |f(z)|$. (see [15] for more detail)

2.2. **Holomorphic function.** In complex analysis, a holomorphic function is a complex-valued function defined and differentiable at every point of an open subset of the complex plane $\mathbb{C}$.

2.3. **Arithmetic function.** Let $f$ be an entire function on $\mathbb{C}$. $f$ is said to be arithmetic if for each non negative integer $n$, $f(n)$ is an integer.

2.4. **Residue.** In complex analysis, the residue is a complex number which describes the behavior of the line integral of a holomorphic function in the neighborhood of a singularity. The residuals are calculated quite easily and, once known, can calculate more complicated curvilinear integrals with the residue theorem.

2.5. **Theorem of residue.** Let $U$ an open simply connected subset of the complex plane $\mathbb{C}$, $\{z_1, \ldots, z_2\}$ a set of points $U$, separate and isolated, and $f$ a function defined and holomorphic on $U \setminus \{z_1, \ldots, z_2\}$.

If $\nabla$ is a rectifiable curve in $U$ which meets no singular points $z_k$ and whose starting point to the end point then:

$$\int_{\nabla} f(z)dz = 2\pi i \sum_{k=1}^{n} Res(f, z_k) Ind_{\nabla}(z_k)$$
where
\[ \text{Ind}_\varphi(z_k) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_k}. \]

3. Theorem of Guelfond

**Theorem 3.1.** [6] Let \( g(z) \) an entire function, and if the \( g(\beta^n) \) are integers, \( n = 1, 2, 3..., (\beta > 1, \text{ integer}) \), if \( g(z) \) verifies:

\[
\ln |g(z)| < \frac{\ln^2 r}{4 \ln \beta} - \frac{1}{2} \ln r - \omega(r), \quad r = |z|, \quad \lim_{r \to +\infty} \omega(r) = \infty,
\]

then \( g(z) \) is a polynom.

To prove this theorem we must show that the quantity:

\[
P_n^k(t) = \frac{(1 - t^n) \ldots (1 - t^{n-k})}{(1 - t) \ldots (1 - t^{k+1})} \quad \text{avec: } n \geq k
\]

is a polynom in \( t \) with integer rational coefficients.

**Proof.** We put:

\[ I_{k+1} = (1 - t) \ldots (1 - t^{k+1}), \]

a) \( t = 1 \) is a root of \( I_{k+1} \) with the order of multiplicity \( k + 1 \).

b) \( t = \exp(2\pi i \frac{p}{q}) \) is a root of \( (1 - t^q), (1 - t^{2q}), \ldots, (1 - t^{\alpha q}) \),

where:

\[
\begin{cases} 
  p \text{ and } q \text{ have no common divisor} \\
  0 \leq p < q \leq k + 1,
\end{cases}
\]

with

\[ 1 \leq \alpha q \leq k + 1 \Rightarrow \frac{1}{q} \leq \alpha \leq \frac{k + 1}{q}. \]

Or the order of the plurality of root \( \exp(2\pi i \frac{p}{q}) \) is \( \alpha = \left\lfloor \frac{k + 1}{q} \right\rfloor \), then the root of the polynom \( I_{k+1} \) are of type \( \exp(2\pi i \frac{p}{q}) \) with multiplicity equal to the order \( \left\lfloor \frac{k + 1}{q} \right\rfloor \).

We denote the polynom:

\( (1 - t^n) \ldots (1 - t^{n-k}) \) by \( I_{n-k} \) with \( n \geq k \),

a) \( t = 1 \) is a root of \( I_{n-k} \) with the order of multiplicity \( k + 1 \).
b) $t = \exp(2\pi i \frac{p}{q})$ is a root of $(1 - t^p), (1 - t^{2p}), \ldots, (1 - t^{\beta q})$, when \( \begin{cases} p \text{ and } q \text{ have no common divisor} \\ 0 \leq p < q \leq k + 1, \end{cases} \)

with

\[ n - k \leq \beta q \leq n \Rightarrow \frac{n - k}{q} \leq \beta \leq \frac{n + 1}{q}; (n \geq k). \]

Or $\beta = \left\lfloor \frac{n + 1}{q} \right\rfloor$ the order of the multiplicity of root $\exp(2\pi i \frac{p}{q})$.

Then $I_{k+1}$ and $I_{n-k}$ are polynomials with roots type $\exp(2\pi i \frac{p}{q})$ was the order of multiplicity equal $\left\lceil \frac{p+1}{q} \right\rceil$ of $I_{k+1}$, and $\left\lfloor \frac{n+1}{q} \right\rfloor$ of $I_{n-k}$.

$I_{k+1}$ divided $I_{n-k}$, with $\left\lceil \frac{k+1}{q} \right\rceil$ (the order of the multiplicity of roots of polynomial $I_{k+1}$) $\leq \left\lfloor \frac{n+1}{q} \right\rfloor$ (the order of the multiplicity of roots of polynomial $I_{n-k}$).

Then

\[ p_n^k(t) = \frac{(1 - t^n) \cdots (1 - t^{n-k})}{(1 - t) \cdots (1 - t^{k+1})}; (n \geq k), \]

is a polynomial, with coefficients are integers because the coefficient of the highest degree of the denominator $((1 - t) \cdots (1 - t^{k+1}))$ equal to unity.

4. **Holomorphic Functions with Complex Variables**

**Theorem 4.1.** Let $g$ an holomorphic function on $\mathbb{C}^2$, with $g(\beta^n, \beta^m)$ are integers, \(\forall n, m \in \mathbb{N} (\beta, \text{integer greater than unity})\) and $g(z_1, z_2)$ verifies the inequality:

\[ \ln |g(z_1, z_2)| < \frac{\ln^2 r_1 r_2}{4 \ln \beta} - \frac{1}{2} \ln r_1 r_2 - \omega(r_1, r_2), \]

or

\[ r_1 = |z_1|, r_2 = |z_2|, \lim_{\min(r_1, r_2) \to +\infty} \omega(r_1, r_2) = \infty. \]

Then $g$ is a polynomial.
Proof. since \( g(z_1, z_2) \) verified the condition:
\[
\ln | g(z_1, z_2) | < \frac{\ln^2 r_1 r_2}{4 \ln \beta} - \frac{1}{2} \ln r_1 r_2 - \omega(r_1, r_2),
\]
therefore
\[
g(z_1, z_2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{n,m} (z_1 - \beta)(z_1^2 - \beta^2) \ldots (z_1 - \beta^n)(z_2 - \beta)(z_2^2 - \beta^2) \ldots (z_2 - \beta^m),
\]
Or
\[
A_{n,m} = \frac{1}{(2\pi i)^2} \int_{c_n} \int_{c_m} \frac{g(z_1, z_2)}{(z_1 - \beta)(z_1 - \beta^2) \ldots (z_1 - \beta^{n+1})(z_2 - \beta)(z_2 - \beta^2) \ldots (z_2 - \beta^{m+1})} \, dz_1 \, dz_2.
\]
Take as an integration contour : \( |z_1| = \beta^{2n} \), \( |z_2| = \beta^{2m} \) after inequality the Cauchy with two variables :
\[
| A_{n,m} | \leq \beta^{2n+2m} \max_{0 \leq \theta_1, \theta_2 \leq 2\pi} | g(\beta^{2n} \exp i\theta_1, \beta^{2m} \exp i\theta_2) | \prod_{k=1}^{n+1} (\beta^{2n} - \beta^k) \prod_{j=1}^{m+1} (\beta^{2m} - \beta^j)
\]
\[
| A_{n,m} | \leq k\beta^{-n(n+3)} \beta^{-m(m+3)} \exp(-w(\beta^{2n}, \beta^{2m})),
\]
but
\[
A_{n,m} = \frac{1}{(2\pi i)^2} \int_{c_n} \int_{c_m} \frac{g(z_1, z_2)}{(z_1 - \beta)(z_1 - \beta^2) \ldots (z_1 - \beta^{n+1})(z_2 - \beta)(z_2 - \beta^2) \ldots (z_2 - \beta^{m+1})} \, dz_1 \, dz_2
\]
\[
= \frac{1}{(2\pi i)^2} \int_{c_n} \left( \frac{1}{2\pi i} \int_{c_m} \frac{g(z_1, z_2)}{(z_2 - \beta)(z_2 - \beta^2) \ldots (z_2 - \beta^{m+1})} \, dz_2 \right) \frac{dz_1}{(z_1 - \beta)(z_1 - \beta^2) \ldots (z_1 - \beta^{n+1})}
\]
(4.1) \[ A_{n,m} = \frac{1}{2\pi i} \int_{c_n} F(z_1) \frac{dz_1}{(z_1 - \beta)(z_1^2 - \beta^2) \ldots (z_1 - \beta^{n+1})}, \]
with \( F(z_1) \)
(4.2) \[ F(z_1) = \frac{1}{2\pi i} \int_{c_m} \frac{g(z_1, z_2)}{(z_2 - \beta)(z_2 - \beta^2) \ldots (z_2 - \beta^{n+1})} \, dz_2. \]
On the other hand the integral giving $A_{n,m}$ equals some residues of the function $F(z_1)$.

(4.3) \[ A_{n,m} = \sum_{s=1}^{n+1} F(\beta^s) \prod_{j=1 \atop s \neq j}^{n+1} (\beta^s - \beta^j) \]

from (4.2)

\[ F(\beta^s) = \frac{1}{2\pi i} \int_{c_m} \frac{g(\beta^s, z_2)}{(z_2 - \beta)(z_2 - \beta^2)\ldots(z_2 - \beta^{m+1})} dz_2, \quad 1 \leq s \leq n + 1 \]

for fixed $s$ : $F(\beta^s) = \frac{1}{2\pi i} \int_{c_m} \frac{G(z_2)}{(z_2 - \beta)(z_2 - \beta^2)\ldots(z_2 - \beta^{m+1})} dz_2,$

with : $g(\beta^s, z_2) = G(z_2)$

\[ G(z_2) = \sum_{m=0}^{\infty} b_m (z_2 - \beta)(z_2 - \beta^2)\ldots(z_2 - \beta^m) \]

or : $b_m = \frac{1}{2\pi i} \int_{c_m} \frac{G(z_2)}{(z_2 - \beta)(z_2 - \beta^2)\ldots(z_2 - \beta^{m+1})} dz_2$

then

$F(\beta^s) = b_m$.

The integral giving $b_m$ equals Some residues of the function $G(z_2)$

\[ b_m = \sum_{k=1}^{m+1} \frac{G(\beta^k)}{\prod_{r=1 \atop r \neq k}^{m+1} (\beta^k - \beta^r)} \]

\[ b_m = \sum_{k=1}^{m+1} \frac{g(\beta^s, \beta^k)}{\prod_{r=1 \atop r \neq k}^{m+1} (\beta^k - \beta^r)} \]
\[(4.4) \quad F(\beta^s) = \sum_{k=1}^{m+1} \frac{g(\beta^s, \beta^k)}{\prod_{\substack{r=1 \atop r \neq k}}^{m+1} (\beta^k - \beta^r)}, \quad (1 \leq s \leq n + 1)\]

replaces (4.4) in (4.3)

\[
A_{n,m} = \sum_{s=1}^{n+1} \frac{F(\beta^s)}{\prod_{j=1 \atop s \neq j}^{n+1} (\beta^s - \beta^j)} \sum_{k=1}^{m+1} \frac{g(\beta^s, \beta^k)}{\prod_{\substack{r=1 \atop r \neq k}}^{m+1} (\beta^k - \beta^r)}
\]

\[
A_{n,m} = \sum_{s=1}^{n+1} \frac{g(\beta^s, \beta^k)}{\prod_{j=1 \atop s \neq j}^{n+1} (\beta^s - \beta^j)} \prod_{\substack{r=1 \atop r \neq k}}^{m+1} (\beta^k - \beta^r)
\]

Then \(A_{n,m}\) is an integer, which implies that all \(A_{n,m}\) vanish for large rank.

Therefore \(g\) is a polynomial. \(\square\)

In general this can result for holomorphic functions has several complex variables.

**Theorem 4.2.** Let \(g\) an holomorphic function on \(\mathbb{C}^d\),

with: \(g(\beta^{n_1}, ..., \beta^{n_d})\) are integers, \(\forall n_1, ..., n_d \in \mathbb{N}^d\) (integer greater than unity),

and \(g(z_1, z_2, ..., z_d)\) verifies the inequality:

\[
\ln |g(z_1, z_2, ..., z_d)| < \frac{\ln^2 r_1 r_2 ... r_d}{4 \ln \beta} - \frac{1}{2} \ln r_1 r_2 ... r_d - \omega(r_1, r_2, ..., r_d)
\]

or

\[
r_j = |z_j|, j = 1, ..., d, \lim_{\min(r_1, r_2, ..., r_d) \to +\infty} \omega(r_1, r_2, ..., r_d) = \infty.
\]
Then $g$ is a polynom.

**Proof.** Since

$$\ln |g(z_1, z_2, ..., z_d)| < \frac{\ln^2 r_1r_2...r_d}{4\ln \beta} - \frac{1}{2} \ln r_1r_2...r_d - \omega(r_1, r_2, ..., r_d),$$

therefore

$$g(z_1, z_2, ..., z_d) = \sum_{n_1=0}^{\infty} ... \sum_{n_d=0}^{\infty} A_{n_1,...,n_d}(z_1 - \beta)(z_1 - \beta^2)...(z_1 - \beta^{n_1})... (z_d - \beta)(z_d - \beta^2)...(z_d - \beta^{n_d}),$$

or

$$A_{n_1,...,n_d} = \frac{1}{(2\pi i)^d} \int_{c_{n_1}} ... \int_{c_{n_d}} \frac{g(z_1, z_2, ..., z_d) \, dz_1...dz_d}{(z_1 - \beta)(z_1 - \beta^2)...(z_1 - \beta^{n_1+1})...(z_d - \beta)(z_d - \beta^2)...(z_d - \beta^{n_d+1})},$$

$$A_{n_1,...,n_d} = \frac{1}{(2\pi i)^d} \int_{c_{n_1}} \frac{1}{(2\pi i)^{d-1}} \int_{c_{n_2}} ... \int_{c_{n_d}} \frac{g(z_1, z_2, ..., z_d) \, dz_1...dz_d}{(z_1 - \beta)(z_1 - \beta^2)...(z_1 - \beta^{n_1+1})...(z_d - \beta)(z_d - \beta^2)...(z_d - \beta^{n_d+1})}dz_1,$$

$$A_{n_1,...,n_d} = \frac{1}{(2\pi i)^d} \int_{c_{n_1}} \frac{1}{(2\pi i)^{d-1}} \int_{c_{n_2}} ... \int_{c_{n_d}} \frac{F(z_1) \, dz_1}{(z_1 - \beta)(z_1 - \beta^2)...(z_1 - \beta^{n_1+1})},$$

with:

$$F(z_1) = \frac{1}{(2\pi i)^{d-1}} \int_{c_{n_2}} ... \int_{c_{n_d}} \frac{g(z_1, z_2, ..., z_d) \, dz_2...dz_d}{(z_2 - \beta)(z_2 - \beta^2)...(z_2 - \beta^{n_2+1})...(z_d - \beta)(z_d - \beta^2)...(z_d - \beta^{n_d+1})},$$

$A_{n_1,...,n_d}$ equals some residues of the function $F(z_1)$. 

HOLOMORPHIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES 289
\[ \begin{align*}
(4.5) \quad & A_{n_1, \ldots, n_d} = \sum_{s=1}^{n_1+1} \frac{F(\beta^s)}{\prod_{j=1}^{n_1+1} (\beta^s - \beta^j)}, \quad 1 \leq s \leq n_1 + 1, \\
& \text{or} \\
F(\beta^s) &= \frac{1}{(2\pi i)^{d-1}} \\
& \int_{c_{n_2}} \cdots \int_{c_{n_d}} \frac{g(\beta^s, z_2, \ldots, z_d)dz_2 \ldots dz_d}{(z_2 - \beta)(z_2 - \beta^2) \ldots (z_2 - \beta^{n_2+1}) \ldots (z_d - \beta)(z_d - \beta^2) \ldots (z_d - \beta^{n_d+1})}, \\
& \quad (1 \leq s \leq n + 1) \\
& \text{for fixed } s: \\
F(\beta^s) &= \frac{1}{(2\pi i)^{d-1}} \int_{c_m} G(z_2, \ldots, z_d)dz_2 \ldots dz_d \\
& \quad \int_{c_m} G(z_2, \ldots, z_d)dz_2 \ldots dz_d \\
& \text{with} \\
g(\beta^s, z_2, \ldots, z_d) &= G(z_2, \ldots, z_d) \\
G(z_2, \ldots, z_d) &= \sum_{n_2=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} b_{n_2, \ldots, n_d} (z_2 - \beta)(z_2 - \beta^2) \ldots (z_2 - \beta^{n_2}) \ldots (z_d - \beta)(z_d - \beta^2) \ldots (z_d - \beta^{n_d}), \\
& \text{or} \\
b_{n_2, \ldots, n_d} &= \frac{1}{(2\pi i)^{d-1}} \int_{c_m} \frac{G(z_2, \ldots, z_d)dz_2 \ldots dz_d}{(z_2 - \beta)(z_2 - \beta^2) \ldots (z_2 - \beta^{n_2+1}) \ldots (z_d - \beta)(z_d - \beta^2) \ldots (z_d - \beta^{n_d+1})}, \\
& \text{then} \\
F(\beta^s) &= b_{n_2, \ldots, n_d}. 
\end{align*} \]
The integral giving \( b_{n_2,...,n_d} \) equals some residues of the function \( G(z_2,...,z_d) \)

\[
b_{n_2,...,n_d} = \sum_{k=1}^{n_2+1} \frac{G(\beta^k, z_3, ..., z_d)}{\prod_{r=1 \atop r \neq k}^{n_2+1} (\beta^k - \beta^r)}
\]

\[
F(\beta^s) = \sum_{k=1}^{n_2+1} \frac{g(\beta^s, \beta^k, z_3, ..., z_d)}{\prod_{r=1 \atop r \neq k}^{n_2+1} (\beta^k - \beta^r)}
\]

from (4.5)

\[
A_{n_1,...,n_d} = \sum_{s=1}^{n_1+1} \sum_{k=1}^{n_2+1} \frac{g(\beta^s, \beta^k, z_3, ..., z_d)}{\prod_{r=1 \atop r \neq k}^{n_2+1} (\beta^k - \beta^r) \prod_{j=1 \atop s \neq j}^{n_1+1} (\beta^s - \beta^j)}
\]

for fixed \( s \) and \( k \)

\[
A_{n_1,...,n_d} = \frac{1}{(2\pi i)^2} \int_{c_{n_1}} \int_{c_{n_2}} \int_{c_{n_3}} \cdots \int_{c_{n_d}} \frac{1}{(2\pi i)^{d-2}} \int_{c_{n_3}} \cdots 
\]

\[
\cdots \int_{c_{n_d}} \frac{g(z_1, z_2, ..., z_d)dz_3...dz_d}{(z_3 - \beta)(z_3 - \beta^2)...(z_3 - \beta^{n_3+1})...(z_d - \beta)(z_d - \beta^2)...(z_d - \beta^{n_d+1})}dz_1dz_2
\]

\[
F(z_1, z_2) = \frac{1}{(2\pi i)^{d-2}} \int_{c_{n_3}} \cdots 
\]

\[
\cdots \int_{c_{n_d}} \frac{g(z_1, z_2, ..., z_d)dz_3...dz_d}{(z_3 - \beta)(z_3 - \beta^2)...(z_3 - \beta^{n_3+1})...(z_d - \beta)(z_d - \beta^2)...(z_d - \beta^{n_d+1})}
\]

\[
A_{n_1,...,n_d} = \frac{1}{(2\pi i)^2} \int_{c_{n_1}} \cdots 
\]

\[
\cdots \int_{c_{n_2}} \frac{F(z_1, z_2)}{(z_1 - \beta)(z_1 - \beta^2)...(z_1 - \beta^{n_1+1})(z_2 - \beta)(z_2 - \beta^2)...(z_2 - \beta^{n_2+1})}dz_1dz_2
\]

\( A_{n_1,...,n_d} \) equals some residues of the function \( F(z_1, z_2) \),
\[ A_{n_1, \ldots, n_d} = \sum_{s=1}^{n_1+1} \sum_{k=1}^{n_2+1} \frac{F(\beta^s, \beta^k)}{\prod_{j=1}^{n_1+1} (\beta^s - \beta^j) \prod_{l=1}^{n_2+1} (\beta^k - \beta^l)}, \quad \begin{cases} 1 \leq s \leq n_1 + 1 \\ 1 \leq k \leq n_2 + 1 \end{cases}, \]

or

\[ F(\beta^s, \beta^k) = \frac{1}{(2\pi i)^{d-2}} \int_{c_{n_3}} \cdots \int_{c_{n_d}} \frac{g(\beta^s, \beta^k, z_d) dz_3 \ldots dz_d}{(z_3 - \beta)(z_3 - \beta^2) \ldots (z_3 - \beta^{n_3+1}) (z_d - \beta)(z_d - \beta^2) \ldots (z_d - \beta^{n_d+1})}, \]

with

\[ \begin{cases} 1 \leq s \leq n_1 + 1 \\ 1 \leq k \leq n_2 + 1 \end{cases}. \]

For \( s \) and \( k \)

\[ F(\beta^s, \beta^k) = \frac{1}{(2\pi i)^{d-2}} \int_{c_{n_3}} \cdots \int_{c_{n_d}} \frac{G(z_3, \ldots, z_d) dz_3 \ldots dz_d}{(z_3 - \beta)(z_3 - \beta^2) \ldots (z_3 - \beta^{n_3+1}) (z_d - \beta)(z_d - \beta^2) \ldots (z_d - \beta^{n_d+1})}, \]

with

\[ g(\beta^s, \beta^k, z_3, \ldots, z_d) = G(z_3, \ldots, z_d) \]

\[ G(z_3, \ldots, z_d) = \sum_{n_3=0}^{\infty} \ldots \sum_{n_d=0}^{\infty} b_{n_3, \ldots, n_d} (z_3 - \beta)(z_3 - \beta^2) \ldots (z_2 - \beta^{n_3})(z_d - \beta)(z_d - \beta^2) \ldots (z_d - \beta^{n_d}), \]

or

\[ b_{n_3, \ldots, n_d} = \frac{1}{(2\pi i)^{d-2}} \int_{c_{n_3}} \cdots \int_{c_{n_d}} \frac{G(z_3, \ldots, z_d) dz_3 \ldots dz_d}{(z_2 - \beta)(z_2 - \beta^2) \ldots (z_2 - \beta^{n_3+1}) (z_d - \beta)(z_d - \beta^2) \ldots (z_d - \beta^{n_d+1})}, \]

then

\[ F(\beta^s, \beta^k) = b_{n_3, \ldots, n_d}. \]
The integral giving $b_{n_3, \ldots, n_d}$ equals some residues of the function $G(z_3, \ldots, z_d)$

\[
b_{n_3, \ldots, n_d} = \sum_{c=1}^{n_3+1} \frac{G(\beta^c, z_4, \ldots, z_d)}{\prod_{c=1, c \neq m}^{n_3+1} (\beta^c - \beta^m)}
\]

\[
F(\beta^s, \beta^k) = \sum_{c=1}^{n_3+1} \frac{g(\beta^s, \beta^k, \beta^c, z_4, \ldots, z_d)}{\prod_{c=1, c \neq m}^{n_3+1} (\beta^c - \beta^m)}.
\]

From (4.6)

\[
A_{n_1, \ldots, n_d} = \sum_{s=1}^{n_1+1} \sum_{k=1}^{n_2+1} \sum_{c=1}^{n_3+1} \frac{g(\beta^s, \beta^k, \beta^c, z_4, \ldots, z_d)}{\prod_{c=1, c \neq m}^{n_3+1} (\beta^c - \beta^m)} \left\{ \begin{array}{l}
1 \leq s \leq n_1 + 1 \\
1 \leq k \leq n_2 + 1
\end{array} \right.
\]

then for fixed all the $s_i$ ($1 \leq i \leq d - 1, 1 \leq s_i \leq n_i + 1$)

\[
A_{n_1, \ldots, n_d} = \sum_{s_1=1}^{n_1+1} \sum_{s_2=1}^{n_2+1} \sum_{s_d=1}^{n_d+1} \frac{g(\beta^{s_1}, \beta^{s_2}, \ldots, \beta^{s_d})}{\prod_{s_1 \neq j}^{n_1+1} (\beta^{s_1} - \beta^j) \prod_{s_2 \neq j}^{n_2+1} (\beta^{s_2} - \beta^j) \cdots \prod_{s_d \neq j}^{n_d+1} (\beta^{s_d} - \beta^j)}
\]

$A_{n_1, \ldots, n_d}$ is an integer, which implies that all $A_{n_1, \ldots, n_d}$ vanish for large rank.

Therefore $g$ is a polynom. \(\square\)

**Corollary 4.1.** Let $F$ the holomorphic function on $\mathbb{C}^d$, such that $F(\beta^{n_1}, \ldots, \beta^{n_d})$ are integers, $\forall n_1, \ldots, n_d \in \mathbb{N}^d(\beta, \text{integer greater than unity})$,

and $F(z_1, z_2, \ldots, z_d)$ satisfies the inequality:

\[
\ln | F(z_1, z_2, \ldots, z_d) | < \frac{\ln^2 r_1 r_2 \cdots r_d}{4 \ln \beta} - \frac{1}{2} \ln r_1 r_2 \cdots r_d - \omega(r_1, r_2, \ldots, r_d)
\]

or

\[
r_j = |z_j|, \; j = 1, \ldots, d, \lim_{\min(r_1, r_2, \ldots, r_d) \to +\infty} \omega(r_1, r_2, \ldots, r_d) = \infty.
\]
If $F(\mathbb{N}^d) = 0$, then $F \equiv 0$ on $\mathbb{C}^d$.

Proof. For Theorem (4.2), $F$ is a polynom and $F(\mathbb{N}^d) = 0$, therefore all the coefficients of polynom are zero.

Then $F \equiv 0$ on $\mathbb{C}^d$. □

References


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