DECOMPOSITIONS OF CONTINUITY VIA GRILLS

AHMAD AL-OMARI AND TAKASHI NOIRI

Abstract. In this paper, we introduce the notions of $G$-$\alpha$-open sets, $G$-semi-open sets and $G$-$\beta$-open sets in grill topological spaces and investigate their properties. Furthermore, by using these sets we obtain new decompositions of continuity.

1. Introduction

The idea of grills on a topological space was first introduced by Choquet [4]. The concept of grills has shown to be a powerful supporting and useful tool like nets and filters, for getting a deeper insight into further studying some topological notions such as proximity spaces, closure spaces and the theory of compactifications and extension problems of different kinds (see [2], [3], [11] for details). In [10], Roy and Mukherjee defined and studied a typical topology associated rather naturally to the existing topology and a grill on a given topological space. Quite recently, Hatir and Jafari [5] have defined new classes of sets in a grill topological space and obtained a new decomposition of continuity in terms of grills. In this paper, we introduce and investigate the notions of $G$-$\alpha$-open sets, $G$-semi-open sets and $G$-$\beta$-open sets in grill topological spaces. We define grill $\alpha$-continuous functions to obtain decompositions of continuity.

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2. Preliminaries

Let \((X, \tau)\) be a topological space with no separation properties assumed. For a subset \(A\) of a topological space \((X, \tau)\), \(Cl(A)\) and \(Int(A)\) denote the closure and the interior of \(A\) in \((X, \tau)\), respectively. The power set of \(X\) will be denoted by \(\mathcal{P}(X)\).

The definition of grill on a topological space, as given by Choquet [4], goes as follows: A non-null collection \(\mathcal{G}\) of subsets of a topological spaces \(X\) is said to be a grill on \(X\) if

1. \(\phi \notin \mathcal{G}\),
2. \(A \in \mathcal{G}\) and \(A \subseteq B\) implies that \(B \in \mathcal{G}\),
3. \(A, B \subseteq X\) and \(A \cup B \in \mathcal{G}\) implies that \(A \in \mathcal{G}\) or \(B \in \mathcal{G}\).

For example let \(R\) be the set of all real numbers consider a subset \(\mathcal{G} = \{A \subseteq R : m(A) \neq 0\}\), where \(m(A)\) is the Lebesgue measure of \(A\), then \(\mathcal{G}\) is a grill. For any point \(x\) of a topological space \((X, \tau)\), \(\tau(x)\) denotes the collection of all open neighborhoods of \(x\).

**Definition 2.1.** [10] Let \((X, \tau)\) be a topological space and \(\mathcal{G}\) be a grill on \(X\). A mapping \(\Phi : \mathcal{P}(X) \to \mathcal{P}(X)\) is defined as follows:

\[\Phi(A) = \Phi_G(A, \tau) = \{x \in X : A \cap U \in \mathcal{G} \text{ for all } U \in \tau(x)\}\]

for each \(A \in \mathcal{P}(X)\). The mapping \(\Phi\) is called the operator associated with the grill \(\mathcal{G}\) and the topology \(\tau\).

**Proposition 2.1.** [10] Let \((X, \tau)\) be a topological space and \(\mathcal{G}\) be a grill on \(X\). Then for all \(A, B \subseteq X\):

1. \(A \subseteq B\) implies that \(\Phi(A) \subseteq \Phi(B)\),
2. \(\Phi(A \cup B) = \Phi(A) \cup \Phi(B)\),
3. \(\Phi(\Phi(A)) \subseteq \Phi(\Phi(A)) = Cl(\Phi(A)) \subseteq Cl(A)\).

Let \(\mathcal{G}\) be a grill on a space \(X\). Then we define a map \(\Psi : \mathcal{P}(X) \to \mathcal{P}(X)\) by \(\Psi(A) = A \cup \Phi(A)\) for all \(A \in \mathcal{P}(X)\). The map \(\Psi\) is a Kuratowski closure axiom. Corresponding to a grill \(\mathcal{G}\) on a topological space \((X, \tau)\), there exists a unique topology
τ_{G} on X given by τ_{G} = \{ U \subseteq X : \Psi(X - U) = X - U \}, where for any A \subseteq X, \Psi(A) = A \cup \Phi(A) = τ_{G}-Cl(A). For any grill G on a topological space (X, τ), τ \subseteq τ_{G}. If (X, τ) is a topological space with a grill G on X, then we call it a grill topological space and denote it by (X, τ, G).

Example 2.1. [10] Let τ denote the cofinite topology on an uncountable set X and let G be the grill of all uncountable subset of X. Then it is clearly τ \setminus \{ \emptyset \} \subseteq G. We show that τ_{G} is the cocountable topology which denoted by τ_{co} on X. If V \in τ_{G}, then V = U - A, where U \in τ and A \notin G implies that (X - U) is finite and A is countable. Now X - V = X \cap (X - V) = X \cap (X - (U \cap (X - A))) = X \cap ((X - U) \cup A) = (X - U) \cup A which is countable and hence V \in τ_{co}. On the other hand if V \in τ_{co} implies that X - V = A \notin G and hence V = X - A, where X \in τ and A \notin G so V \in τ_{G}. Thus τ_{G} = τ_{co}.

Lemma 2.1. [10] For any grill G on a topological space (X, τ), τ \subseteq B(G, τ) \subseteq τ_{G}, where B(G, τ) = \{ V - A : V \in τ and A \notin G \} is an open base for τ_{G}.

Example 2.2. Let (X, τ) be a topological space. If G = \mathcal{P}(X) \setminus \{ \emptyset \}, then τ_{G} = τ. Since for any τ_{G}-basic open set V = X - A with U \in τ and A \notin G, we have A = \emptyset, so that V = U \in τ. Hence by Lemma 2.1 we have in this case τ = B(G, τ) = τ_{G}.

Definition 2.2. A subset A of a topological space X is said to be:

1. α-open [9] if A \subseteq Int(Cl(Int(A))),
2. semi-open [6] if A \subseteq Cl(Int(A)),
3. preopen [8] if A \subseteq Int(Cl(A)),
4. β-open [1] if A \subseteq Cl(Int(Cl(A))).


Definition 3.1. Let (X, τ, G) be a grill topological space. A subset A in X is said to be
(1) $\Phi$-open [5] if $A \subseteq \text{Int}(\Phi(A))$,
(2) $\mathcal{G}$-$\alpha$-open if $A \subseteq \text{Int}(\Psi(\text{Int}(A)))$,
(3) $\mathcal{G}$-preopen [5] if $A \subseteq \text{Int}(\Psi(A))$,
(4) $\mathcal{G}$-semi-open if $A \subseteq \Psi(\text{Int}(A))$,
(5) $\mathcal{G}$-$\beta$-open if $A \subseteq \text{Cl}(\text{Int}(\Psi(A)))$.

The family of all $\mathcal{G}$-$\alpha$-open (resp. $\mathcal{G}$-preopen, $\mathcal{G}$-semi-open, $\mathcal{G}$-$\beta$-open) sets in a grill topological space $(X, \tau, \mathcal{G})$ is denoted by $\mathcal{G}\alpha O(X)$ (resp. $\mathcal{G}PO(X)$, $\mathcal{G}SO(X)$, $\mathcal{G}\beta O(X)$).

Remark 1. For several sets defined above, we have the following implications, where converses of implications need not be true as shown by below examples.

Example 3.1. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$ and the grill
$\mathcal{G} = \{\{a\}, \{b\}, \{a, c\}, \{a, b\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{c, b, d\}, \{a, c, d\}, \{b, c\}, \{b, d\}, \{b, c, d\}, X\}$.

Then
(1) $A = \{b, c, d\}$ is a semi-open set which is not $\mathcal{G}$-semi-open.
(2) $A = \{b, c\}$ is a $\mathcal{G}$-$\beta$-open set which is not $\mathcal{G}$-semi-open.
(3) $B = \{a, b\}$ is a $G$-semi-open set which is not preopen and hence it is not $G$-preopen.

(4) $C = \{a, b, c\}$ is a $G$-$\alpha$-open set which is not open.

**Example 3.2.** Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$ and the grill $G = \{\{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{a, b, d\}, X\}$. Then $A = \{a, c, d\}$ is an $\alpha$-open set and a $G$-$\beta$-open set which is not $G$-preopen.

**Example 3.3.** Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$ and the grill $G = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, c\}, X\}$. Then

(1) $A = \{a, c\}$ is a $\beta$-open set which is not $G$-$\beta$-open.

(2) $B = \{a, b\}$ is a $G$-preopen set which is not $G$-semi-open.

**Proposition 3.1.** For a subset of a grill topological space $(X, \tau, G)$, the following properties are hold:

(1) Every $G$-$\alpha$-open set is $\alpha$-open.

(2) Every $G$-semi-open set is semi-open.

(3) Every $G$-$\beta$-open set is $\beta$-open.

**Theorem 3.1.** Let $A$ be a subset of a grill topological space $(X, \tau, G)$. Then the following properties hold:

(1) A subset $A$ of $X$ is $G$-$\alpha$-open if and only if it is $G$-semi-open and $G$-pre-open,

(2) If $A$ is $G$-semi-open, then $A$ is $G$-$\beta$-open.

(3) If $A$ is $G$-preopen, then $A$ is $G$-$\beta$-open.

**Proof.**

(1) **Necessity.** This is obvious.

**Sufficiency.** Let $A$ be $G$-semi-open and $G$-pre-open. Then we have $A \subseteq \text{Int}(\Psi(A)) \subseteq \text{Int}(\Psi(\text{Int}(A))) \subseteq \text{Int}(\Psi(\text{Int}(A)))$. This shows that $A$ is $G$-$\alpha$-open.

(2) Since $A$ is $G$-semi-open and $\tau \subseteq \tau_G$, we have $A \subseteq \Psi(\text{Int}(A)) \subseteq \text{Cl}(\text{Int}(A)) \subseteq \text{Cl}(\text{Int}(\Psi(A)))$. This shows that $A$ is $G$-$\beta$-open.

(3) The proof is obvious.  \[\square\]
Theorem 3.2. A subset $A$ of a grill topological space $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-open if and only if $\Psi(A) = \Psi(\text{Int}(A))$.

Theorem 3.3. A subset $A$ of a grill topological space $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-open if and only if there exists $U \in \tau$ such that $U \subseteq A \subseteq \Psi(U)$.

Proof. Let $A$ be $\mathcal{G}$-semi-open, then $A \subseteq \Psi(\text{Int}(A))$. Take $U = \text{Int}(A)$. Then we have $U \subseteq A \subseteq \Psi(U)$. Conversely, let $U \subseteq A \subseteq \Psi(U)$ for some $U \in \tau$. Since $U \subseteq A$, we have $U \subseteq \text{Int}(A)$ and hence $\Psi(U) \subseteq \Psi(\text{Int}(A))$. Thus we obtain $A \subseteq \Psi(\text{Int}(A))$. □

Theorem 3.4. If $A$ is a $\mathcal{G}$-semi-open set in a grill topological space $(X, \tau, \mathcal{G})$ and $A \subseteq B \subseteq \Psi(A)$, then $B$ is $\mathcal{G}$-semi-open in $(X, \tau, \mathcal{G})$.

Proof. Since $A$ be $\mathcal{G}$-semi-open, there exists an open set $U$ of $X$ such that $U \subseteq A \subseteq \Psi(U)$. Then we have $U \subseteq A \subseteq B \subseteq \Psi(A) \subseteq \Psi(\Psi(U)) = \Psi(U)$ and hence $U \subseteq B \subseteq \Psi(U)$. By Theorem 3.3, we obtain that $B$ is $\mathcal{G}$-semi-open in $(X, \tau, \mathcal{G})$. □

Lemma 3.1. [10] Let $(X, \tau)$ be a topological space and $\mathcal{G}$ be a grill on $X$. If $U \in \tau$, then $U \cap \Phi(A) = U \cap \Phi(U \cap A)$ for any $A \subseteq X$.

Lemma 3.2. Let $A$ be a subset of a grill topological space $(X, \tau, \mathcal{G})$. If $U \in \tau$, then $U \cap \Psi(A) \subseteq \Psi(U \cap A)$.

Proof. Since $U \in \tau$, by Lemma 3.1 we obtain $U \cap \Psi(A) = U \cap (A \cup \Phi(A)) = (U \cap A) \cup (U \cap \Phi(A)) \subseteq (U \cap A) \cup \Phi(U \cap A) = \Psi(U \cap A)$. □

Proposition 3.2. Let $(X, \tau, \mathcal{G})$ be a grill topological space.

1. If $V \in \mathcal{G}SO(X)$ and $A \in \mathcal{G} \alpha O(X)$, then $V \cap A \in \mathcal{G}SO(X)$.
2. If $V \in \mathcal{G} PO(X)$ and $A \in \mathcal{G} \alpha O(X)$, then $V \cap A \in \mathcal{G} PO(X)$.

Proof. (1) Let $V \in \mathcal{G}SO(X)$ and $A \in \mathcal{G} \alpha O(X)$. By using Lemma 3.2 we obtain
$V \cap A \subseteq \Psi(\text{Int}(V)) \cap \text{Int}(\Psi(\text{Int}(A)))$
\[\subseteq \Psi[\text{Int}(V) \cap \text{Int}(\Psi(\text{Int}(A)))]\]
\[\subseteq \Psi[\text{Int}(V) \cap \Psi(\text{Int}(A))]
\[\subseteq \Psi[\text{Int}(V) \cap \text{Int}(A)]\]
\[\subseteq \Psi[\text{Int}(V \cap A)].\]

This shows that $V \cap A \in \mathcal{G}SO(X)$.

(2) Let $V \in \mathcal{G}PO(X)$ and $A \in \mathcal{G}oaO(X)$. By using Lemma 3.2 we obtain

$V \cap A \subseteq \text{Int}(\Psi(V)) \cap \text{Int}(\Psi(\text{Int}(A)))$
\[= \text{Int}[\text{Int}(\Psi(V)) \cap \Psi(\text{Int}(A))]
\[\subseteq \text{Int}[\Psi[\text{Int}(\Psi(V)) \cap \text{Int}(A)]]\]
\[\subseteq \text{Int}[\Psi[\Psi(V) \cap \text{Int}(A)]]\]
\[\subseteq \text{Int}[\Psi[\text{Int}(V) \cap \text{Int}(A)]]\]
\[\subseteq \text{Int}[\Psi[\text{Int}(V \cap A)]]\].

This shows that $V \cap A \in \mathcal{G}PO(X)$. □

**Corollary 3.1.** Let $(X, \tau, \mathcal{G})$ be a grill topological space.

(1) If $V \in \mathcal{G}SO(X)$ and $A \in \tau$, then $V \cap A \in \mathcal{G}SO(X)$.

(2) If $V \in \mathcal{G}PO(X)$ and $A \in \tau$, then $V \cap A \in \mathcal{G}PO(X)$.

**Proposition 3.3.** Let $(X, \tau, \mathcal{G})$ be a grill topological space.

(1) If $A, B \in \mathcal{G}oaO(X)$, then $A \cap B \in \mathcal{G}oaO(X)$.

(2) If $A_i \in \mathcal{G}oaO(X)$ for each $i \in I$, then $\bigcup_{i \in I} A_i \in \mathcal{G}oaO(X)$. 
Proof. (1) Let \( A, B \in G_\alpha O(X) \). By Theorem 3.1 \( A \) is \( G \)-semi-open and \( G \)-pre-open and by Proposition 3.2 \( A \cap B \) is \( G \)-semi-open and \( G \)-pre-open. Therefore, \( A \cap B \in G_\alpha O(X) \).

(2) Let \( A_i \in G_\alpha O(X) \) for each \( i \in I \). Then, we have

\[
A_i \subseteq \text{Int}(\Psi(\text{Int}(A_i))) \subseteq \text{Int}(\Psi(\text{Int}(\bigcup_{i \in I} A_i)))
\]

This shows that \( \bigcup_{i \in I} A_i \in G_\alpha O(X) \).

\( \square \)

Corollary 3.2. Let \((X, \tau, G)\) be a grill topological space. Then the family \( G_\alpha O(X) \) is a topology for \( X \) such that \( \tau \subseteq G_\alpha O(X) \subseteq \tau^\alpha \), where \( \tau^\alpha \) denotes the family of \( \alpha \)-open sets of \( X \).

Proof. Since \( \phi, X \in G_\alpha O(X) \), this is an immediate consequence of Propositions 3.1 and 3.3.

\( \square \)

Example 3.4. Let \( X = \{a, b, c, d\} \), \( \tau = \{\phi, X, \{a\}, \{a, b\}\} \) and the grill

\( G = \{\{a, b\}, \{a, b, c\}, \{a, b, d\}, X\} \). Then

\( \tau^\alpha = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, c, d\}, \{a, b, d\}, \{a, c, d\}\} \) and

\( G_\alpha O(X) = \{\phi, X, \{a\}, \{a, b\}, \{a, b, d\}, \{a, b, c\}\} \) and hence \( \tau \subsetneq G_\alpha O(X) \subsetneq \tau^\alpha \).

Remark 2. (1) The minimal grill is \( G = \{X\} \) in any a topological space \((X, \tau)\).

(2) The maximal grill is \( G = \mathcal{P}(X) \setminus \{\phi\} \) in any a topological space \((X, \tau)\).

The proofs of the following three corollary is straightforward, hence it is omitted.

Corollary 3.3. Let \((X, \tau, G)\) be a grill topological space and \( A \) a subset of \( X \). If \( G = \mathcal{P}(X) \setminus \{\phi\} \). Then the following hold:

(1) \( A \) is \( G \)-\( \alpha \)-open if and only if \( A \) is \( \alpha \)-open.

(2) \( A \) is \( G \)-preopen if and only if \( A \) is preopen.

(3) \( A \) is \( G \)-semi-open if and only if \( A \) is semi-open.

(4) \( A \) is \( G \)-\( \beta \)-open if and only if \( A \) is \( \beta \)-open.
Let \((X, \tau, \mathcal{G})\) be a grill topological space. If \(\mathcal{G} = \{X\}\), then \(\Phi(A) = \emptyset\) for any subset \(A\) of \(X\) and \(\Psi(A) = \tau_\mathcal{G} \cdot Cl(A) = A\) and hence \(\tau_\mathcal{G} = \tau_{dis}\), where \(\tau_{dis}\) is the discrete topology on \(X\).

**Corollary 3.4.** Let \((X, \tau, \mathcal{G})\) be a grill topological space and \(A\) a subset of \(X\). If \(\Phi(A) = \emptyset\) for any subset \(A\) of \(X\) and \(\Psi(A) = \tau_\mathcal{G} \cdot Cl(A) = A\) and hence \(\tau_\mathcal{G} = \tau_{dis}\), where \(\tau_{dis}\) is the discrete topology on \(X\).

**Corollary 3.5.** Let \((X, \tau, \mathcal{G})\) be a grill topological space and \(A\) a subset of \(X\). If \(\Phi(A) = Cl(\text{Int}(Cl(A)))\) for any subset \(A\) of \(X\). Then the following hold:

1. \(A\) is \(\mathcal{G}\)-\(\alpha\)-open if and only if \(A\) is \(\alpha\)-open.
2. \(A\) is \(\mathcal{G}\)-preopen if and only if \(A\) is \(\beta\)-open.

Recall that \((X, \tau)\) is called submaximal if every dense subset of \(X\) is open.

**Lemma 3.3.** [7] If \((X, \tau)\) is submaximal, then \(PO(X, \tau) = \tau\).

**Corollary 3.6.** If \((X, \tau)\) is submaximal, then for any grill \(\mathcal{G}\) on \(X\), \(\tau = \alpha O(X) = PO(X, \tau) = \mathcal{G}PO(X) = \mathcal{G}O(X)\).

**Theorem 3.5.** Let \((X, \tau, \mathcal{G})\) be a grill topological space and \(A, B\) subsets of \(X\). If \(U_\alpha \in \mathcal{G}SO(X, \tau)\) for each \(\alpha \in \Delta\), then \(\cup \{U_\alpha : \alpha \in \Delta\} \in \mathcal{G}SO(X, \tau)\).

**Proof.** Since \(U_\alpha \in \mathcal{G}SO(X, \tau)\), we have \(U_\alpha \subseteq \Psi(\text{Int}(U_\alpha))\) for each \(\alpha \in \Delta\). Thus we obtain \(U_\alpha \subseteq \Psi(\text{Int}(U_\alpha)) \subseteq \Psi(\text{Int}(\cup_{\alpha \in \Delta} U_\alpha))\) and hence \(\cup_{\alpha \in \Delta} U_\alpha \subseteq \Psi(\text{Int}(\cup_{\alpha \in \Delta}(U_\alpha)))\). This shows that \(\cup \{U_\alpha : \alpha \in \Delta\} \in \mathcal{G}SO(X, \tau)\). \(\square\)

**Definition 3.2.** A subset \(F\) of a grill topological space \((X, \tau, \mathcal{G})\) is said to be \(\mathcal{G}\)-semi-closed (resp. \(\mathcal{G}\)-preclosed) if its complement is \(\mathcal{G}\)-semi-open (resp. \(\mathcal{G}\)-preopen).

**Theorem 3.6.** If a subset \(A\) of a grill topological space \((X, \tau, \mathcal{G})\) is \(\mathcal{G}\)-semi-closed, then \(\text{Int}(\Psi(A)) \subseteq A\).
Theorem 3.7. If a subset $A$ of a grill topological space $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-preclosed, then $\Psi(\text{Int}(A)) \subseteq A$.

Definition 3.3. Let $(X, \tau, \mathcal{G})$ be a grill topological space. A subset $A$ in $X$ is called

(1) a $g_1$-set if $\text{Int}(\Psi(\text{Int}(A))) = \text{Int}(A)$,
(2) a $g_2$-set if $\Psi(\text{Int}(A)) = \text{Int}(A)$.

Definition 3.4. Let $(X, \tau, \mathcal{G})$ be a grill topological space. A subset $A$ in $X$ is called

(1) a $G_1$-set if $A = U \cap V$, where $U \in \tau$ and $V$ is a $g_1$-set,
(2) a $G_2$-set if $A = U \cap V$, where $U \in \tau$ and $V$ is a $g_2$-set.

Theorem 3.8. Let $(X, \tau, \mathcal{G})$ be a grill topological space. For a subset $A$ of $X$, the following conditions are equivalent:

(1) $A$ is open;
(2) $A$ is $\mathcal{G}$-$\alpha$-open and a $G_1$-set;
(3) $A$ is $\mathcal{G}$-semi-open and a $G_2$-set.

Proof. (1) $\Rightarrow$ (2) Let $A$ be any open set. Then we have $A = \text{Int}(A) \subseteq \text{Int}(\Psi(\text{Int}(A)))$. Therefore $A$ is $\mathcal{G}$-$\alpha$-open and because $X$ is a $g_1$-set, hence $A$ is a $G_1$-set.

(2) $\Rightarrow$ (1) Let $A$ be $\mathcal{G}$-$\alpha$-open and a $G_1$-set. Let $A = U \cap C$, where $U$ is open and $\text{Int}(\Psi(\text{Int}(C))) = \text{Int}(C)$. Since $A$ is a $\mathcal{G}$-$\alpha$-open set, we have

$$U \cap C \subseteq \text{Int}(\Psi(\text{Int}(U \cap C)))$$

$$= \text{Int}(\Psi(\text{Int}(U) \cap \text{Int}(C)))$$

$$= \text{Int}(\Psi(U \cap \text{Int}(C)))$$

$$\subseteq \text{Int}(\Psi(U) \cap \Psi(\text{Int}(C)))$$

$$= \text{Int}(\Psi(U)) \cap \text{Int}(\Psi(\text{Int}(C)))$$

$$= \text{Int}(\Psi(U)) \cap \text{Int}(C).$$
Since \( U \subseteq \text{Int}(\Psi(U)) \), we have
\[
U \cap C = (U \cap C) \cap U \subseteq \text{Int}(\Psi(U)) \cap \text{Int}(C) \cap U = U \cap \text{Int}(C) = \text{Int}(U \cap C).
\]
Therefore, \( A = U \cap C \) is an open set.

(1) \( \Rightarrow \) (3) This is obvious, because \( X \) is a \( g_2 \)-set, then \( A \) is a \( G_2 \)-set.

(3) \( \Rightarrow \) (1) Suppose that \( A \) is \( G \)-semi-open and a \( G_2 \)-set. Let \( A = U \cap C \), where \( U \) is open and \( \Psi(\text{Int}(C)) = \text{Int}(C) \). Since \( A \) is a \( G \)-semi-open set, we have
\[
\begin{align*}
U \cap C &\subseteq \Psi(\text{Int}(U \cap C)) \\
&= \Psi(\text{Int}(U) \cap \text{Int}(C)) \\
&= \Psi(U \cap \text{Int}(C)) \\
&\subseteq \Psi(U) \cap \Psi(\text{Int}(C)) \\
&= \Psi(U) \cap \text{Int}(C).
\end{align*}
\]
Since \( U \subseteq \Psi(U) \), we have
\[
U \cap C = (U \cap C) \cap U \subseteq \Psi(U) \cap \text{Int}(C) \cap U = U \cap \text{Int}(C) = \text{Int}(U \cap C).
\]
Therefore, \( A = U \cap C \) is an open set.

The notion of \( G_\alpha \)-openness (resp. \( G \)-semi-openness) is different from that of \( G_1 \)-sets (resp. \( G_2 \)-sets).

Remark 3. (1) In Example 3.1, \( A = \{a, b\} \) is a \( g_1 \)-set and hence a \( G_1 \)-set but it is not \( G_\alpha \)-open. And \( B = \{a, b, c\} \) is \( G \)-open but it is not a \( G_1 \)-set.

(2) In Example 3.2, \( A = \{a, c, d\} \) is a \( g_2 \)-set and hence a \( G_2 \)-set but it is not \( G \)-semi-open.

(3) In Example 3.1, \( B = \{a, b, c\} \) is \( G \)-semi-open but it is not a \( G_2 \)-set.

4. Decompositions of Continuity

Definition 4.1. A function \( f : (X, \tau, \mathcal{G}) \to (Y, \sigma) \) is said to be grill \( \alpha \)-continuous (resp. grill semi-continuous, grill pre-continuous [5]) if the inverse image of each open set of \( Y \) is \( \mathcal{G}_\alpha \)-open (resp. \( \mathcal{G} \)-semi-open, \( \mathcal{G} \)-preopen).
Theorem 4.1. For a function $f : (X, \tau, G) \to (Y, \sigma)$, the following properties are equivalent:

1. $f$ is grill $\alpha$-continuous;
2. For each $x \in X$ and each $V \in \sigma$ containing $f(x)$, there exists $W \in G_{\alpha O}(X)$ containing $x$ such that $f(W) \subseteq V$;
3. The inverse image of each closed set in $Y$ is $G$-$\alpha$-closed;
4. $Cl(Int_G(Cl(f^{-1}(B)))) \subseteq f^{-1}(Cl(B))$ for each $B \subseteq Y$;
5. $f(Cl(Int_G(Cl(A)))) \subseteq Cl(f(A))$ for each $A \subseteq X$.

Proof. The implications follow easily from the definitions. \qed

Corollary 4.1. Let $f : (X, \tau, G) \to (Y, \sigma)$ be grill $\alpha$-continuous, then

1. $f(\Psi(U)) \subseteq Cl(f(U))$ for each $U \in GPO(X)$.
2. $\Psi(f^{-1}(V)) \subseteq f^{-1}(Cl(V))$ for each $V \in GPO(Y)$.

Theorem 4.2. A function $f : (X, \tau, G) \to (Y, \sigma)$ is grill $\alpha$-continuous if and only if the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is grill $\alpha$-continuous.

Definition 4.2. A function $f : (X, \tau, G) \to (Y, \sigma, H)$ is said to be grill irresolute if $f^{-1}(V)$ is $G$-semi-open in $(X, \tau, G)$ for each $G$-semi-open $V$ of $(Y, \sigma, H)$.

Remark 4. It is obvious that continuity implies grill semi-continuity and grill semi-continuity implies semi-continuity.

Theorem 4.3. For a function $f : (X, \tau, G) \to (Y, \sigma)$, the following are equivalent:

1. $f$ is grill semi-continuous.
2. For each $x \in X$ and each $V \in \sigma$ containing $f(x)$, there exists $U \in GSO(X)$ containing $x$ such that $f(U) \subseteq V$.
3. The inverse image of each closed set in $Y$ is $G$-semi-closed.

Theorem 4.4. Let $f : (X, \tau, G) \to (Y, \sigma, H)$ be grill semi-continuous and $f^{-1}(\Psi(V)) \subseteq \Psi(f^{-1}(V))$ for each $V \in \sigma$. Then $f$ is grill irresolute.
Proof. Let $B$ be any $\mathcal{G}$-semi-open set of $(Y, \sigma, \mathcal{H})$. By Theorem 3.3, there exists $V \in \sigma$ such that $V \subseteq B \subseteq \Psi(V)$. Therefore, we have $f^{-1}(V) \subseteq f^{-1}(B) \subseteq f^{-1}(\Psi(V)) \subseteq \Psi(f^{-1}(V))$. Since $f$ is grill semi-continuous and $V \in \sigma$, $f^{-1}(V) \in \mathcal{G}SO(X)$ and hence by Theorem 3.4, $f^{-1}(B)$ is a $\mathcal{G}$-semi-open set of $(X, \tau, \mathcal{G})$. This shows that $f$ is grill irresolute. □

Theorem 4.5. A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is grill semi-continuous if and only if the graph function $g : X \to X \times Y$ is grill semi-continuous.

Theorem 4.6. A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is grill $\alpha$-continuous if and only if it is grill semi-continuous and grill pre-continuous.

Proof. This is an immediate consequence of Theorem 3.1. □

Theorem 4.7. A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is grill $\alpha$-continuous if and only if $f : (X, \mathcal{G}O(X)) \to (Y, \sigma)$ is continuous.

Proof. This is an immediate consequence of Corollary 3.2. □

Definition 4.3. A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is said to be $G_1$-continuous (resp. $G_2$-continuous) if the inverse image of each open set of $Y$ is $G_1$-open (resp. $G_2$-open) in $(X, \tau, \mathcal{G})$.

Theorem 4.8. Let $(X, \tau, \mathcal{G})$ be a grill topological space. For a function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$, the following conditions are equivalent:

1. $f$ is continuous;
2. $f$ is grill $\alpha$-continuous and $G_1$-continuous;
3. $f$ is grill semi-continuous and $G_2$-continuous.

Proof. This is an immediate consequence of Theorem 3.8. □

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