THE PERIODS OF THE PELL P-ORBITS OF POLYHEDRAL AND CENTRO-POLYHEDRAL GROUPS

ÖMÜR DEVEÇİ(1), MERVE AKDENIZ(2) AND YEŞİM AKÜZÜM(3)

Abstract. In this paper, we define the Pell p-orbit of a finitely generated group and then we obtain the lengths of the periods and the basic periods of the Pell p-orbits of the finite polyhedral groups and centro-polyhedral groups.

1. INTRODUCTION AND PRELIMINARIES

The study of recurrence sequences in groups began with the earlier work of Wall [3] where the ordinary Fibonacci sequences in cyclic groups were investigated. The concept extended to some special linear recurrence sequences by several authors; see for example, [1, 2, 5, 6, 8, 9, 10, 11, 13, 14, 15, 16]. In [12] extended the theory to the generalized Pell p-sequences. In this paper, we examine the behaviour of the periods and basic periods of the Pell p-orbits of the polyhedral groups \((n, 2, 2), (2, n, 2), (2, 2, n), (2, 3, 3), (2, 3, 4), (2, 2, 5)\) and the centro-polyhedral groups \(\langle -2, n, 2 \rangle, \langle 2, n, -2 \rangle, \langle n, -2, 2 \rangle, \langle n, 2, -2 \rangle, \langle 2, -2, n \rangle, \langle -2, 2, n \rangle\) for \(n > 2\).

In [4], the generalized Pell \((p, i)\) numbers was defined as follows: for \(p (p = 1, 2, \cdots), n > p + 1\) and \(0 \leq i \leq p,\)

\[(1.1) \quad P_p^{(i)}(n) = 2P_p^{(i)}(n - 1) + P_p^{(i)}(n - p - 1),\]

2000 Mathematics Subject Classification. 11K31, 11B50, 20F05, 20D60.

Key words and phrases. Pell p-orbit, Period, Group.

Copyright © Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

Received: Nov. 25, 2014 Accepted: Jan. 3, 2017.
with initial conditions $P_{p}^{(i)}(1) = \cdots = P_{p}^{(i)}(i) = 0$ and $P_{p}^{(i)}(i + 1) = \cdots = P_{p}^{(i)}(p + 1) = 1$.

Note that if $i = 0$, the initial conditions are $P_{p}^{(i)}(1) = P_{p}^{(i)}(2) = \cdots = P_{p}^{(i)}(p + 1) = 1$.

A sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called the period of the sequence. For example, the sequence $a, b, c, d, a, b, c, d, a, b, c, d, \ldots$ is periodic after the initial element $a$ and has period 3. A sequence is simply periodic with period $k$ if the first $k$ elements in the sequence form a repeating subsequence. For example, the sequence $a, b, c, d, a, b, c, d, a, b, c, d, \ldots$ is simply periodic with period 4.

Reducing the generalized Pell $(p, p)$-sequence $\{P_{p}^{(p)}(n)\}$ by a modulus $m$, we can get repeating sequence, denoted by

$$\{P_{p}^{(p,m)}(n)\} = \{P_{p}^{(p,m)}(1), P_{p}^{(p,m)}(2), \ldots, P_{p}^{(p,m)}(p), P_{p}^{(p,m)}(p + 1), \ldots, P_{p}^{(p,m)}(i), \ldots\}$$

where $P_{p}^{(p,m)}(i) = P_{p}^{(p)}(i) \pmod{m}$. Also, it has the same recurrence relation as in (1.1) (Deveci et al.). ([12, p.3]).

**Theorem 1.1.** (Deveci et al.). ([12, Theorem 2.1, p.3]) $\{P_{p}^{(p,m)}(n)\}$ is a simply periodic sequence.

The notation $h_{p}^{(p,m)}(m)$ is used for the smallest period of the sequence $\{P_{p}^{(p,m)}(n)\}$ (Deveci et al.). ([12, p.3]).

Let $G$ be a finite $j$-generator group and let

$$X = \{(x_0, x_1, \ldots, x_{j-1}) \in G \times G \times \cdots \times G | < \{x_0, x_1, \ldots, x_{j-1}\} >= G\}.$$

We call $(x_0, x_1, \ldots, x_{j-1})$ a generating $j$-tuple for $G$. 
Definition 1.1. (Deveci et al.) ([12, Definition 3.4, p.6]) A generalized Pell $p$-sequence ($p \geq 2$) in a finite group is a sequence of group elements $x_0, x_1, \ldots, x_n, \ldots$ for which, given an initial (seed) set $x_0, \ldots, x_{j-1}$, $(j \geq 1)$ each element is defined by

$$x_n = \begin{cases} x_0 (x_{n-1})^2 & \text{for } j \leq n < p+1, \\ x_{n-p-1} (x_{n-1})^2 & \text{for } n \geq p+1. \end{cases}$$

It is required that the initial (seed) set $x_0, \ldots, x_{j-1}$ of the group elements sequence generates the group, thus, forcing the generalized Pell $p$-sequence to reflect the structure of the group.

The generalized Pell $p$-sequence of a group generated by $x_0, \ldots, x_{j-1}$ is denoted by $Q_p(G; x_0, x_1, \ldots, x_{j-1})$.

Theorem 1.2. (Deveci et al.) ([12, Theorem 3.1, p.7]) A generalized Pell $p$-sequence in a finite group is simply periodic.

In (Deveci et al.) ([12, p.7]), the period of the generalized Pell $p$-sequence $Q_p(G; x_0, x_1, \ldots, x_{j-1})$ had been denoted by $\text{Per}Q_p(G; x_0, x_1, \ldots, x_{j-1})$.

Definition 1.2. (Deveci et al.) ([12, Definition 3.5, p.8]) Let $G$ be a finite $j$-generator groups. For a $j$-tuple $(x_0, x_1, \ldots, x_{j-1}) \in X$ the basic generalized Pell $p$-sequence $Q_p(G; x_0, x_1, \ldots, x_{j-1})$, $(p \geq 2, p+1 \geq j)$ of the basic period $m$ is a sequence of group elements $a_0, a_1, a_2, \ldots, a_n, \ldots$ for which, given an initial (seed) set $a_0 = x_0$, $a_1 = x_1$, $a_2 = x_2$, $\ldots$, $a_{j-1} = x_{j-1}$, each element is defined by

$$a_n = \begin{cases} a_0 (a_{n-1})^2 & \text{for } j \leq n < p+1, \\ a_{n-p-1} (a_{n-1})^2 & \text{for } n \geq p+1 \end{cases}$$

where $m \geq 1$ is the least integer with

$$a_0 = a_m \theta, \ a_1 = a_{m+1} \theta, \ a_2 = a_{m+2} \theta, \ldots, \ a_p = a_{m+p} \theta.$$
for some $\theta \in \text{Aut} G$. Since $G$ is a finite $j$-generator group and $a_m, a_{m+1}, \ldots, a_{m+j-1}$ generate $G$, it follows that $\theta$ is uniquely determined. The basic generalized Pell $p$-sequence $Q^{(p)}(G; x_0, x_1, \ldots, x_{j-1})$ is finite containing $m$ element.

Also, in (Deveci et al.).([12, p.8]), the basic period of the basic generalized Pell $p$-sequence $Q^{(p)}(G; x_0, x_1, \ldots, x_{j-1})$ had been denoted by $BQ^{(p)}(G; x_0, x_1, \ldots, x_{j-1})$.

**Definition 1.3.** The polyhedral group $(l, m, n)$ for $l, m, n > 1$, is defined by the presentation

$$\langle x, y, z : x^l = y^m = z^n = xyz = 1 \rangle.$$ 

For the generating pair $(x, y)$, the polyhedral group $(l, m, n)$ have the presentations

$$\langle x, y : x^l = y^m = (xy)^n = 1 \rangle$$

and

$$\langle x, y : x^l = y^m = (xy)^{-n} = 1 \rangle,$$

where $l, m, n > 1$.

The polyhedral group $(l, m, n)$ is finite if and only if the number $k = lmn \left( \frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1 \right) = mn+nl+lm-lmn$ is positive. Its order is $2lmn/k$ (Coxeter and Moser).([7, p. 67-68]).

In this paper, we consider polyhedral groups as 3-generator groups.

**Definition 1.4.** The centro-polyhedral group $\langle l, m, n \rangle$, for $l, m, n \in Z$ is defined by the presentation

$$\langle x, y, z : x^l = y^m = z^n = xyz \rangle.$$ 

For detailed information about these groups, see(Coxeter and Moser).([7, p. 70-71]).
2. MAIN RESULTS AND PROOFS

**Definition 2.1.** For a finitely generated group $G = \langle A \rangle$, where $A = \{a_1, a_2, \ldots, a_{p+1}\}$ such that $p \geq 2$, the Pell $p$-orbit of $G$ with respect to the generating set $A$, written $P^p_A(G)$ is the sequence $x_0 = a_1$, $x_1 = a_2, \ldots, x_p = a_{p+1}$, $x_{n+p} = (x_{n-1})(x_{n+p-1})^2$, $n \geq 1$. The length of the period of the sequence is called the Pell $p$-length of $G$ with respect to the generating set $A$, written $LP^p_A(G)$. Also, we denote the length of basic period of this sequence by $\overline{LP}^p_A(G)$, which is called the basic length of $G$ with respect to the generating set $A$.

Firstly, we consider the Pell $p$-lengths and the basic Pell $p$-lengths of the finite polyhedral groups by the following Theorems.

**Theorem 2.1.** Let $G$ be any of the polyhedral groups $(n, 2, 2)$, $(2, n, 2)$ and $(2, 2, n)$, where $n \geq 3$. Then

$$LP^2_{\{x, y, z\}}(G) = \begin{cases} \frac{3n}{2}, & n \text{ is even} \\ 3n, & n \text{ is odd} \end{cases}$$

and $\overline{LP}^2_{\{x, y, z\}}(G) = 3$.

**Proof.** Let us consider the group $(n, 2, 2)$. The orbit $P^2_{\{x, y, z\}}((n, 2, 2))$ is

$x, y, z, x, yx^2, z, x, yx^4, z, x, yx^6, z, x, yx^8, z, x, yx^{10}, z, \ldots$

This sequence can be said to form layers of length three. Using the above, the sequence becomes:

$x_0 = x$, $x_1 = y$, $x_2 = z$,

$x_3 = x$, $x_4 = yx^2$, $x_5 = z$,

$x_6 = x$, $x_7 = yx^4$, $x_8 = z$,

$x_{3i} = x$, $x_{3i+1} = yx^{2i}$, $x_{3i+2} = z$, \ldots

So, we need the smallest $i \in N$ such that $2i = nv_1$ for $v_1 \in N$. 
If \( n \) is even, then \( i = \frac{n}{2} \). Thus, \( LP_{\{x,y,z\}}^2 ((n, 2, 2)) = \frac{3n}{2} \) and \( \overline{LP}_{\{x,y,z\}}^2 ((n, 2, 2)) = 3 \) since \( x\theta = x, y\theta = yx^{-2} \) and \( z\theta = z \) where \( \theta \) is an outer automorphism of order \( \frac{n}{2} \).

If \( n \) is odd, then \( n = i \). Thus, \( LP_{\{x,y,z\}}^2 ((n, 2, 2)) = 3n \) and \( \overline{LP}_{\{x,y,z\}}^2 ((n, 2, 2)) = 3 \) since \( x\theta = x, y\theta = yx^{-2} \) and \( z\theta = z \) where \( \theta \) is an outer automorphism of order \( n \).

The proofs for the groups \((2, n, 2)\) and \((2, 2, n)\) are similar to the above and are omitted. \(\square\)

**Theorem 2.2.**

i) \( LP_{\{x,y,z\}}^2 ((2, 3, 3)) = \overline{LP}_{\{x,y,z\}}^2 ((2, 3, 3)) = 65. \)

ii) \( LP_{\{x,y,z\}}^2 ((2, 3, 4)) = \overline{LP}_{\{x,y,z\}}^2 ((2, 3, 4)) = 27. \)

iii) \( LP_{\{x,y,z\}}^2 ((2, 3, 5)) = \overline{LP}_{\{x,y,z\}}^2 ((2, 3, 5)) = 175. \)

**Proof.** The orbit \( P_{\{x,y,z\}}^2 ((2, 3, 3)) \) is

\[
\{x, y, z, y, 1, z, yxy, xyx, xy, yxy^2, yxy^3, yxy^4, yxy^5, yxy^6, \ldots, x, y, z, y, 1, z, yxy, xyx, xy, yxy^2, yxy^3, yxy^4, yxy^5, yxy^6, \ldots\},
\]

which has period 65. Also, \( \overline{LP}_{\{x,y,z\}}^2 ((2, 3, 3)) = 65 \) since \( x\theta = x, y\theta = y \) and \( z\theta = z \) where \( \theta \) is identity automorphism.

The proofs of the cases ii and iii are similar to the above and are omitted. \(\square\)

Now we give the Pell p-lengths and the basic Pell p-lengths of some centro polyhedral groups by the following Theorem.

**Theorem 2.3.** Let \( G \) be any of the centro-polyhedral groups \(\langle -2, n, 2 \rangle, \langle 2, n, -2 \rangle, \langle n, -2, 2 \rangle, \langle n, 2, -2 \rangle, \langle 2, -2, n \rangle \) and \(\langle -2, 2, n \rangle\), where \( n \geq 3 \). Then

\[
LP_{\{x,y,z\}}^2 (G) = \begin{cases} 
\frac{n}{2} \cdot h_2^2 (4 (n - 1)) & n \text{ is even,} \\
 n \cdot h_2^2 (4 (n - 1)) & n \text{ is odd}
\end{cases}
\]

and \( \overline{LP}_{\{x,y,z\}}^2 (G) = h_2^2 (4 (n - 1)) \),

where \( h_2^2 (4 (n - 1)) \) denotes the smallest period of the sequence \( \{P_2^{(2,4(n-1))} (n)\} \).
Proof. Let us consider the group \(\langle -2, n, 2 \rangle\). It is clear that the centro polyhedral group \(\langle -2, n, 2 \rangle\) is defined by the presentation

\[
\langle x, y, z : x^{-2} = y^n = z^2 = xyz \rangle.
\]

writing \(x^{-2} = y^n = z^2 = xyz = s\), we find that \(|s| = \frac{4n}{4n + 1} - 1 = n - 1\). Thus we obtain \(|\langle -2, n, 2 \rangle| = 4n (n - 1)\), \(|x| = |z| = 4 (n - 1)\) and \(|y| = 2n (n - 1)\). Also note that \(z^2\) is central element of the group \(\langle -2, n, 2 \rangle\).

If \(n\) is a positive even integer, then the orbit \(P^2_\{x,y,z\} \langle -2, n, 2 \rangle\) becomes:

\[
x_0 = x, \ x_1 = y, \ x_2 = z, \\
x_{h_2^j(4(n-1))} = x, \ x_{h_2^j(4(n-1))} + 1 = y, \ x_{h_2^j(4(n-1))} + 2 = y z^1, \ h_2^j(4(n-1)), \\
x_{h_2^j(4(n-1))} = x, \ x_{h_2^j(4(n-1))} + 1 = y, \ x_{h_2^j(4(n-1))} + 2 = y z^2, \ h_2^j(4(n-1)), \ldots,
\]

where \(k_1 \in N\) be such that \((k_1, \frac{n}{2}) = 1\). Since \(|y| = 2n (n - 1)\), we need the smallest \(i \in N\) such that \(k_1 \cdot 4 (n - 1) i = 2n (n - 1) v_2\) for \(v_2 \in N\). Then, we obtain \(i = \frac{n}{2}\) for \(v_2 = k_1\) since \(n\) is a positive even integer. Thus, \(LP^2_\{x,y,z\} \langle -2, n, 2 \rangle\) and \(LP^2_\{x,y,z\} \langle -2, n, 2 \rangle = h_2^2 (4 (n - 1))\) since \(x \theta = x, \ y \theta = y\) and \(z \theta = y z^1\).

where \(\theta\) is an outer automorphism of order \(\frac{n}{2}\) and \(t_1 \in N\) such that \((t_1, \frac{n}{2}) = 1\).

If \(n\) is a positive odd integer, then the orbit \(P^2_\{x,y,z\} \langle -2, n, 2 \rangle\) becomes:

\[
x_0 = x, \ x_1 = y, \ x_2 = z, \\
x_{h_2^j(4(n-1))} = x, \ x_{h_2^j(4(n-1))} + 1 = y, \ x_{h_2^j(4(n-1))} + 2 = z y^1, \ h_2^j(4(n-1)), \\
x_{h_2^j(4(n-1))} = x, \ x_{h_2^j(4(n-1))} + 1 = y, \ x_{h_2^j(4(n-1))} + 2 = z y^2, \ h_2^j(4(n-1)), \ldots,
\]

where \(k_2 \in N\) be such that \((k_2, n) = 1\). Since \(|y| = 2n (n - 1)\), we need the smallest \(i \in N\) such that \(k_2 \cdot 4 (n - 1) i = 2n (n - 1) v_3\) for \(v_3 \in N\). Then, we obtain \(i = n\) for \(k_2 = v_3\) since \(n\) is a positive odd integer. Thus, \(LP^2_\{x,y,z\} \langle -2, n, 2 \rangle = n \cdot h_2^2 (4 (n - 1))\) and \(LP^2_\{x,y,z\} \langle -2, n, 2 \rangle = h_2^2 (4 (n - 1))\) since \(x \theta = x, \ y \theta = y\) and \(z \theta = y z^1\).

where \(\theta\) is an outer automorphism of order \(n\) and \(t_2 \in N\) such that \((t_2, n) = 1\).
The proofs for the groups \(\langle 2, n, -2 \rangle, \langle n, -2, 2 \rangle, \langle n, 2, -2 \rangle, \langle 2, -2, n \rangle\) and \(\langle -2, 2, n \rangle\) are similar to the above and are omitted. 

All necessary calculations were carried out on the computer using the GAP computational algebra system, see (The GAP group).[17]

**Acknowledgement.**

The authors thank the referees for their valuable suggestions which improved the presentation of the paper. This Project was supported by the Commission for the Scientific Research Projects of Kafkas University. The Project number. 2014-FEF-34.

**REFERENCES**


1,2,3DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND LETTERS, KAFKAS UNIVERSITY, 36100 KARS, TURKEY

E-mail address: odeveci36@hotmail.com; merveakdeniz55@gmail.com; yesim_036@hotmail.com