ON D-METACOMPACTNESS IN BITOPOLITICAL SPACES

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Abstract. In this paper we define pairwise D-metacompact spaces and study their properties and their relations with other topological spaces. Several examples are discussed and many well known theorems are generalized concerning metacompact spaces.

1. Introduction

In 1963, Kelly [7] introduced the notion of a bitopological space, i.e. a triple $(X, \tau_1, \tau_2)$ where $X$ is a set and $\tau_1, \tau_2$ are two topologies on $X$. He also defined pairwise regular ($P-$regular), pairwise normal ($P-$normal), and obtained generalization of several standard results such as Urysohn’s lemma and Tietze extension theorem.

Several authors have since considered the problem of defining compactness for such spaces, see Kim [8], Fletcher, Hoyle and Patty [5]. In 1969, Fletcher et. al. [5] gave the definitions of $\tau_1 \tau_2-$open and $P-$open covers in bitopological spaces. A cover $\tilde{U}$ of the bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_1 \tau_2-$open if $\tilde{U} \subset \tau_1 \cup \tau_2$, if in addition, $\tilde{U}$ contains at least one non-empty member of $\tau_1$ and at least one non-empty member

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of $\tau_2$, it is called $P$–open. Also they defined the concept of $P$–compact space as follows: A bitopological space $(X, \tau_1, \tau_2)$ is called $P$–compact if every $P$–open cover of the space $(X, \tau_1, \tau_2)$ has a finite subcover. While in 1972, Datta [3] defined $S$–compact space as follows: A bitopological space $(X, \tau_1, \tau_2)$ is called $S$–compact if every $\tau_1\tau_2$–open cover of the space $(X, \tau_1, \tau_2)$ has a finite subcover. In 1969, Birsan [1] gave the following definitions: A bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_1$–compact with respect to $\tau_2$ if for each $\tau_1$–open cover of $X$, there is a finite $\tau_2$–open subcover. A bitopological space $(X, \tau_1, \tau_2)$ is called $B$–compact if it is $\tau_1$–compact with respect to $\tau_2$ and $\tau_2$–compact with respect to $\tau_1$. In 1975, Cooke and Reilly [2] discussed the relations between these definitions. In 1983 Fora and Hdieb [6] introduced the definition of $P$–Lindel"{o}f, $S$–Lindel"{o}f, $B$–Lindel"{o}f spaces in analogue manner. They also gave the definitions of certain types of functions as follows: A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called $P$–continuous ($P$–open, $P$–closed, $P$–homeomorphism, respectively), if both functions $f_1 : (X, \tau_1) \rightarrow (Y, \sigma_1)$ and $f_2 : (X, \tau_2) \rightarrow (Y, \sigma_2)$ are continuous (open, closed, homeomorphism, respectively).

In this paper we introduce the notion of $D$–metacompact spaces in bitopological spaces, and derive some related results. When $(X, \tau_1, \tau_2)$ has the property $Q$ this means that both $(X, \tau_1)$ and $(X, \tau_2)$ have this property. For instance a bitopological space $(X, \tau_1, \tau_2)$ is called metacompact, if both $(X, \tau_1)$ and $(X, \tau_2)$ are metacompact spaces.

We will use the letters $P$–, $S$– to denote the pairwise and semi, respectively, e.g. $P$–metacompact stands for pairwise metacompact, and similarly, one can define $P$–compact, $P$–Lindel"{o}f, ... etc. Also, $S$–metacompact stands for semi metacompact, and similarly, one can define $S$–Lindel"{o}f, ... etc.

Also we will use the letters $P - D$–, $S - D$– to denote the pairwise–$D$ and semi–$D$, respectively, e.g. $P - D$–metacompact stands for pairwise $D$–metacompact, $S - D$–metacompact stands for semi $D$–metacompact.
$\tau_i$—closure, $\tau_i$—interior of a set $A$ will be denoted by $CL_iA$, $Int_iA$ respectively. The product of $\tau_1$ and $\tau_2$ will be denoted by $\tau_1 \times \tau_2$.

Let $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ denote the set of all real numbers, integer numbers, natural numbers, and rational numbers, respectively. Let $\tau_{dis}$, $\tau_{ind}$, $\tau_u$, $\tau_{coc}$, $\tau_{cof}$, $\tau_l$, $\tau_r$ denote the discrete, the indiscrete usual, Sorgenfrey line, ccountable, cofinite, left-ray, and right-ray topologies, respectively. Let $\omega_0$ and $\omega_1$ denote the cardinal numbers of $\mathbb{Z}$ and $\mathbb{R}$, respectively.

2. Pairwise D-Metacompact Spaces

In this section, we will introduce the concept of D-metacompactness in bitopological spaces, and introduce some of their properties, and relate it to other spaces.

Let us recall known definitions which will be used in the sequel.

**Definition 2.1.** A subset $A$ of topological space $(X, \tau)$ is called a $D$—set if there are two open sets $U$ and $V$ such that $U \neq X$ and $A = U - V$. In this case we say that $A$ is a $D$—set generated by $U$ and $V$.

Observe that every open set $U$ different from $X$ is a $D$—set if $A = U$ and $V = \phi$.

**Definition 2.2.** A cover $\tilde{D} = \{D_\alpha : \alpha \in \Delta\}$ of a topological space $(X, \tau)$ is said to be $D$—cover if each $D_\alpha$ is a $D$—set for all $\alpha \in \Delta$.

In a bitopological space $(X, \tau_1, \tau_2)$, the $D$—sets generated by open sets in $(X, \tau_i)$ are called $\tau_i$—$D$—sets denoted by $D_{\tau_i}$.

A cover $\tilde{D}$ of the bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_1\tau_2$—$D$—cover, if $\tilde{D} \subset D_{\tau_1} \cup D_{\tau_2}$. If in addition, $\tilde{D}$ contains at least one non-empty $\tau_1$—$D$ set and at least one non-empty $\tau_2$—$D$ set, it is called $P$—$D$—cover.

A pairwise $D$—cover $\tilde{V}$ of a bitopological space $(X, \tau_1, \tau_2)$ is called parallel refinement of pairwise $D$—cover $\tilde{U}$ of $X$ if each $\tau_i$—$D$—set of $\tilde{V}$ is contained in some $\tau_i$—$D$—set of $\tilde{U}$ ($i = 1, 2$).
A pairwise $D$–cover $\tilde{U}$ of the bitopological space $(X, \tau_1, \tau_2)$ is called pairwise point finite if each $x \in X$ is contained in a finite number of $\tau_1$–$D$–members of $\tilde{U}$, or it contained in a finite number of $\tau_2$–$D$–members of $\tilde{U}$.

It is clear that every $P$–open cover with proper subsets is a $P$–$D$–cover, but the converse needs not be true. In the bitopological space $(\mathbb{R}, \tau_{cof}, \tau_{dis})$, $\{\{x\} : x \in \mathbb{R}\}$ is a $P$–$D$–cover that is not $P$–open cover.

**Definition 2.3.** A bitopological space $(X, \tau_1, \tau_2)$ is called $P$–$D$–metacompact if every $P$–$D$–cover of the space $(X, \tau_1, \tau_2)$ has a pairwise point finite parallel refinement.

A bitopological space $(X, \tau_1, \tau_2)$ is called $S$–$D$–metacompact if every $\tau_1 \tau_2$–$D$–cover of the space $(X, \tau_1, \tau_2)$ has a pairwise point finite parallel refinement.

A bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_1$–$D$–metacompact with respect to $\tau_2$–$D$–cover if each $\tau_1$–$D$–cover of $X$ has a point finite $\tau_2$–$D$–parallel refinement.

A bitopological space $(X, \tau_1, \tau_2)$ is called $B$–$D$–metacompact, if it is $\tau_1$–$D$–metacompact with respect to $\tau_2$–$D$–metacompact with respect to $\tau_1$.

**Theorem 2.1.** Every $P$–$D$–metacompact space $(X, \tau_1, \tau_2)$ is $P$–metacompact.

*Proof.* Let $\tilde{U} = \{U_{\alpha} : \alpha \in \Delta\}$ be a $P$–open cover of $(X, \tau_1, \tau_2)$. Then $\tilde{U}$ is a $P$–$D$–cover, so it has a pairwise point finite parallel refinement. Hence the result. □

Recall that: A space $(X, \tau)$ is said to be locally indiscrete if every open set is clopen.

**Definition 2.4.** A bitopological space $(X, \tau_1, \tau_2)$ is called locally indiscrete if every $\tau_i$–open set is $\tau_i$–clopen ($i = 1, 2$).

In a locally indiscrete space $(X, \tau_1, \tau_2)$, every $\tau_i$–$D$–set is clopen.

The converse of above theorem is not true as we will see in the following example.
Example 2.1. Let $\tau_1 = \{ U \subset \mathbb{R} : Q \subset U \} \cup \{ \phi \}$ and

$\tau_2 = \{ U \subset \mathbb{R} : \mathbb{R} - U \text{ is uncountable} \} \cup \{ \phi \}$. Then $(\mathbb{R}, \tau_1, \tau_2)$ is $P$-metacompact which is not $P - D$-metacompact.

Since the $P - D$-cover $\{ \{ x \} : x \in \mathbb{R} - Q \} \cup \{ (-n, n) : n \in \mathbb{N} \}$ has no point finite parallel refinement.

The following theorem shows that the converse of the above theorem can be true under extra conditions.

Theorem 2.2. Every locally indiscrete $P$-metacompact bitopological space $(X, \tau_1, \tau_2)$ is $P - D$-metacompact.

Proof. Let $\tilde{U}$ be a $P - D$-cover of $(X, \tau_1, \tau_2)$. Then $\tilde{U} = \{ U_\alpha : \alpha \in \Delta \} \cup \{ V_\beta : \beta \in \Gamma \}$, where $U_\alpha \in D_{\tau_1}$ for each $\alpha \in \Delta$ and $V_\beta \in D_{\tau_2}$ for each $\beta \in \Gamma$. Since $(X, \tau_1, \tau_2)$ is locally indiscrete, $U_\alpha$ is clopen set for each $\alpha \in \Delta$ and $V_\beta$ is clopen set for each $\beta \in \Gamma$. Hence $\tilde{U} = \{ U_\alpha : \alpha \in \Delta \} \cup \{ V_\beta : \beta \in \Gamma \}$ is a $P$-open cover, so it has a pairwise point finite parallel refinement. Hence the result. \qed

It is clear that the locally indiscrete bitopological space $(\mathbb{R}, \tau_{ind}, \tau_{dis})$ is $P - D$-metacompact, since it is $P$-metacompact.

Theorem 2.3. The bitopological space $(X, \tau_1, \tau_2)$ is $S - D$-metacompact if and only if it is $D$-metacompact and $P - D$-metacompact.

Proof. $\implies$ Assume that $(X, \tau_1, \tau_2)$ is $S - D$-metacompact. Let $\tilde{U}$ be a $P - D$-cover of $X$. Then $\tilde{U}$ is a $\tau_1 \tau_2 - D$-cover of the space $(X, \tau_1, \tau_2)$. Since $(X, \tau_1, \tau_2)$ is $S - D$-metacompact, $\tilde{U}$ has a pairwise point finite parallel refinement. Hence $(X, \tau_1, \tau_2)$ is $P - D$-metacompact. Also any $\tau_1 - D$-cover or $\tau_2 - D$-cover of $(X, \tau_1, \tau_2)$ is a $\tau_1 \tau_2 - D$-cover. Hence $(X, \tau_1)$ and $(X, \tau_2)$ are $D$-metacompact. So $(X, \tau_1, \tau_2)$ is $D$-metacompact.
Assume that \((X, \tau_1, \tau_2)\) is \(D\)-metacompact and \(P - D\)-metacompact. Let \(\tilde{U}\) be a \(\tau_1 \tau_2 - D\)-cover of \((X, \tau_1, \tau_2)\). If \(\tilde{U}\) is a \(P - D\)-cover, then the result follows. If \(\tilde{U}\) is not \(P - D\)-cover, then it is \(\tau_1 - D\)-cover or \(\tau_2 - D\)-cover of \((X, \tau_1, \tau_2)\). Since \((X, \tau_1, \tau_2)\) is \(D\)-metacompact, \(\tilde{U}\) has a pairwise point finite parallel refinement. So \((X, \tau_1, \tau_2)\) is \(S - D\)-metacompact. \(\square\)

In a bitopological space \((X, \tau_1, \tau_2)\), the least upper bound topology of \(\tau_1\) and \(\tau_2\) is the smallest topology that contains \(\tau_1 \cup \tau_2\).

The following theorems can be proved easily.

**Theorem 2.4.** The bitopological space \((X, \tau_1, \tau_2)\) is \(S - D\)-metacompact if and only if \((X, \tau)\) is \(D\)-metacompact, where \(\tau\) is the least upper bound topology of \(\tau_1\) and \(\tau_2\).

**Corollary 2.1.** The bitopological space \((X, \tau_1, \tau_2)\) is \(S\)-metacompact if and only if \((X, \tau)\) is metacompact, where \(\tau\) is the least upper bound topology of \(\tau_1\) and \(\tau_2\).

**Theorem 2.5.** If a bitopological space \((X, \tau_1, \tau_2)\) is \(B - D\)-metacompact, then both \((X, \tau_1)\) and \((X, \tau_2)\) must be \(D\)-metacompact spaces.

**Corollary 2.2.** If a bitopological space \((X, \tau_1, \tau_2)\) is \(B\)-metacompact, then both \((X, \tau_1)\) and \((X, \tau_2)\) must be metacompact spaces.

Recall that a space has a hereditary property \(P\), if every subspace of it has this property.

**Theorem 2.6.** If a bitopological space \((X, \tau_1, \tau_2)\) is hereditary \(D\)-metacompact, then it is \(S - D\)-metacompact.

**Proof.** Let \(\tilde{U}\) be a \(\tau_1 \tau_2 - D\)-cover of \((X, \tau_1, \tau_2)\). Then \(\tilde{U} = \{U_{\alpha} : \alpha \in \Delta\} \cup \{V_{\beta} : \beta \in \Gamma\}\), where \(U_{\alpha}\) is a \(\tau_1 - D\) set for each \(\alpha \in \Delta\) and \(V_{\beta}\) is a \(\tau_2 - D\) set for each \(\beta \in \Gamma\). Let \(U = \bigcup_{\alpha \in \Delta} U_{\alpha}\). If \(U\) is finite, we are done. Else write \(U_{\alpha} = K_{\alpha} - W_{\alpha}\) where \(K_{\alpha}, W_{\alpha} \in \tau_1\).
and $K_\alpha \neq X$. We can write $U_\alpha = K_\alpha \cap U - W_\alpha \cap U$. Let $\Omega = \{\alpha \in \Delta : K_\alpha \cap U = U\}$. So the sets $\{U_\alpha : \alpha \in \Delta - \Omega\}$ is a family of $D-$sets in $U$. If $x \in U - \bigcup_{\alpha \in \Delta - \Omega} U_\alpha$. Let $y \in U - \{x\}$. So there exists an open set $O_x$ in $\tau_1$ such that $x \in O_x \cap U$ and $y \notin O_x$. Let $\hat{U}_{\alpha x} = (O_x \cap K_{\alpha x} \cap U) - W_{\alpha x} \cap U \subseteq U_{\alpha x}$, where $x \in U_{\alpha x} \in \hat{U}$. Now, $\{U_\alpha : \alpha \in \Delta - \Gamma\} \cup \{\hat{U}_{\alpha x} : x \in U - \bigcup_{\alpha \in \Delta - \Omega} U_\alpha\}$ is a $D-$cover for $U$. Similarly, we can do the same for the $\tau_2 - D-$cover for $V = \bigcup_{\beta \in \Gamma} V_\beta$.

Since $U$ is $\tau_1 - D-$metacompact, it has a point finite parallel refinement say $\{U_\alpha^* : \alpha \in \Delta'\}$ and $U = \bigcup_{\alpha \in \Delta'} U_\alpha^*$. Similarly, since $V$ is $\tau_2 - D-$metacompact, it has a point finite parallel refinement say $\{V_\beta^* : \beta \in \Gamma'\}$ and $V = \bigcup_{\beta \in \Gamma'} V_\beta^*$.

Hence, $\{U_\alpha^* : \alpha \in \Delta'\} \cup \{V_\beta^* : \beta \in \Gamma'\}$ is a $\tau_1 \tau_2 - D-$point finite parallel refinement of $\hat{U}$. Hence the result. \hfill \Box

**Definition 2.5.** A bitopological space $(X, \tau_1, \tau_2)$ is called $P - D-$Lindelöf if every $P - D-$cover of the space $(X, \tau_1, \tau_2)$ has a countable subcover.

A bitopological space $(X, \tau_1, \tau_2)$ is called $S - D-$Lindelöf if every $\tau_1 \tau_2 - D-$cover of the space $(X, \tau_1, \tau_2)$ has a countable subcover.

A bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_1 - D-$Lindelöf with respect to $D_{\tau_2}$ if for each $\tau_1 - D-$cover of $X$, there is a countable $\tau_2 - D-$subcover.

A bitopological space $(X, \tau_1, \tau_2)$ is called $B - D-$Lindelöf if it is $\tau_1 - D-$Lindelöf with respect to $D_{\tau_2}$ and $\tau_2 - D-$Lindelöf with respect to $D_{\tau_1}$.

**Example 2.2.** Consider the two topologies $\tau_1$ and $\tau_2$ on $\mathbb{R}$ defined by the basis:

\[ \beta_1 = \{(-\infty, a) : a > 0\} \cup \{\{x\} : x > 0\} \]

\[ \beta_2 = \{(a, \infty) : a < 0\} \cup \{\{x\} : x < 0\} \]

Then $X$ is $P-$metacompact, $P - D-$metacompact but not a $B-$metacompact, since for the $\tau_1-$open cover $\{(-\infty, 2)\} \cup \{\{x\} : x > 1\}$ of $\mathbb{R}$ has no point finite
\( \tau_2 \)-open refinement. So it is not a \( B - D \)-metacompact. It is clear that both \((\mathbb{R}, \tau_1)\) and \((\mathbb{R}, \tau_2)\) are \( D \)-metacompact spaces, so \((\mathbb{R}, \tau_1, \tau_2)\) is \( D \)-metacompact.

Observe that \((\mathbb{R}, \tau_1, \tau_2)\) is \( S \)-metacompact, \( S - D \)-metacompact. On the other hand we have \((\mathbb{R}, \tau_1, \tau_2)\) is countably \( P - D \)-metacompact. Now we observe that \((\mathbb{R}, \tau_1, \tau_2)\) is \( P - \text{Lindel"{o}f} \), \( P - D - \text{Lindel"{o}f} \), it is also clear that \((\mathbb{R}, \tau_1, \tau_2)\) is not \( B - \text{Lindel"{o}f} \), and not \( \text{Lindel"{o}f} \), so it is not \( B - D - \text{Lindel"{o}f} \).

**Definition 2.6.** A subset \( D \) of a bitopological space \((X, \tau_1, \tau_2)\) is called pairwise dense denoted by \((P \text{-dense})\) in \( X \), if \( CL_{\tau_1}D = CL_{\tau_2}D = X \).

A bitopological space \((X, \tau_1, \tau_2)\) is called \( P \)-separable, if it has a \( P \)-dense countable subset \( D \).

**Definition 2.7.** A subset \( A \) of a topological space \((X, \tau)\) is called \( D \)-dense, if for all \( x \in X \) and every \( D \)-set \( D_x \) containing \( x \) we have \( D_x \cap A \neq \emptyset \).

It is clear that every \( D \)-dense set is dense. The converse is not true, since in \((\mathbb{R}, \tau_{cof})\) the set of all irrational numbers \( k = \mathbb{R} - \mathbb{Q} \) is dense but not \( D \)-dense, since \( \{5\} \) is a \( D \)-set and \( k \cap \{5\} = \emptyset \).

**Definition 2.8.** A subset \( A \) of a bitopological space \((X, \tau_1, \tau_2)\) is called pairwise \( D \)-dense, if for all \( x \in X \) and every \( \tau_i - D \)-set \( D_x \) containing \( x \) we have \( D_x \cap A \neq \emptyset \), \((i = 1, 2)\).

A bitopological space \((X, \tau_1, \tau_2)\) is called \( P - D \)-separable, if it has a \( P - D \)-dense countable subset \( D \).

It is clear that the bitopological space \((\mathbb{Z}, \tau_u, \tau_{cof})\) is \( P - D \)-separable.

**Theorem 2.7.** A \( P - D \)-separable, \( P - D \)-metacompact space \((X, \tau_1, \tau_2)\) is \( P - D \)-Lindel"{o}f.
Proof. Let \( \tilde{U} = \{U_\alpha : \alpha \in \Delta\} \) be a \( P - D \)-cover of \( X \). Assume that \( \tilde{U} \) has no countable subcover. Let \( \tilde{V} = \{V_\beta : \beta \in \Gamma\} \) be a point finite parallel refinement of \( \tilde{U} \). Let \( D \) be a countable \( D \)-dense subset of \( X \). Then \( V_\beta \cap D \neq \varnothing \) for each \( \beta \in \Gamma \). Thus \( \tilde{V} \) is countable set. So, we may write \( \tilde{V} = \{V_i : i \in \mathbb{N}\} \). But for each \( i \in \mathbb{N} \), we have \( V_i \subseteq U_{\alpha_i} \) for some \( \alpha_i \in \Delta \). Thus \( X = \bigcup_{i \in \mathbb{N}} V_i \subseteq \bigcup_{i \in \mathbb{N}} U_{\alpha_i} \subseteq X \). Hence \( \{U_{\alpha_i} : i \in \mathbb{N}\} \) is a countable subcover of \( \tilde{U} \). \( \square \)

Since every \( P \)-open cover is \( P - D \)-cover, The following corollaries are easily proved.

**Corollary 2.3.** A \( P - D \)-separable, \( P - D \)-metacompact space \((X, \tau_1, \tau_2)\) is \( P - \text{Lindelöf} \).

**Corollary 2.4.** A \( P \)-separable, \( P \)-metacompact space \((X, \tau_1, \tau_2)\)

is \( P - \text{Lindelöf} \).

The last corollary can be also found in [11].

**Example 2.3.** (1) The bitopological space \((\mathbb{N}, \tau_{\text{dis}}, \tau_{\text{ind}})\) is \( P - D \)-metacompact, \( S \)-\( D \)-metacompact, not \( B - D \)-metacompact, \( D \)-metacompact space since \((\mathbb{N}, \tau_{\text{dis}})\) and \((\mathbb{N}, \tau_{\text{ind}})\) are \( D \)-metacompact. It is also countably \( P - D \)-metacompact, not countably \( B - D \)-metacompact. On the other hand \((\mathbb{N}, \tau_{\text{dis}}, \tau_{\text{ind}})\)

is \( P - D \)-Lindelöf, \( P \)-separable, and so, it is \( P - D \)-separable.

(2) The bitopological space \((\mathbb{R}, \tau_{\text{dis}}, \tau_u)\) is \( P - D \)-metacompact,

\( B - D \)-metacompact, \( D \)-metacompact space since \((\mathbb{R}, \tau_{\text{dis}})\) and \((\mathbb{R}, \tau_u)\) are \( D \)-metacompact. It is also countably \( P - D \)-metacompact, countably \( B - D \)-metacompact. It is not \( P \)-separable, so it is not \( P - D \)-separable. It is clear that \((\mathbb{R}, \tau_{\text{dis}}, \tau_u)\) is neither \( P \)-Lindelöf nor \( P \)-compact. It is not \( P \)-countably compact. So \((\mathbb{R}, \tau_{\text{dis}}, \tau_u)\) is neither \( P - D \)-Lindelöf nor \( P - D \)-compact. Also it is not \( P - D \)-countably compact.

**Theorem 2.8.** Every \( P - D \)-Lindelöf countably \( P - D \)-metacompact space \((X, \tau_1, \tau_2)\)

is \( P - D \)-metacompact.
Proof. Let $\tilde{U} = \{ U_\alpha : \alpha \in \Delta \}$ be a $P-D-$cover of $X$. Since $X$ is $P-D-$Lindelöf, $\tilde{U}$ has a countable subcover $\tilde{V} = \{ V_\alpha \}_{i=1}^\infty$. Since $X$ is countably $P-D-$metacompact, $\tilde{V}$ has a point finite parallel refinement $\tilde{W}$ of $\tilde{U}$. Hence $(X, \tau_1, \tau_2)$ is $P-D-$metacompact.

Since every $P-$open cover is $P-D-$cover, the following corollaries are easily proved.

**Corollary 2.5.** Every $P-D-$Lindelöf countably $P-D-$metacompact space $(X, \tau_1, \tau_2)$ is $P-$metacompact space.

**Corollary 2.6.** Every $P-$Lindelöf countably $P-D-$metacompact space $(X, \tau_1, \tau_2)$ is $P-$metacompact.

**Corollary 2.7.** Every $P-$Lindelöf countably $P-$metacompact space $(X, \tau_1, \tau_2)$ is $P-$metacompact.

The last corollary can be also found in [11].

It is clear that the bitopological space $(\mathbb{Z}, \tau_{dis}, \tau_{ind})$ is $P-D-$metacompact, since it is countably $P-D-$metacompact and $P-D-$Lindelöf.

**Theorem 2.9.** Every $P-D-$metalindelöf, countably $P-D-$metacompact space $(X, \tau_1, \tau_2)$ is $P-D-$metacompact space.

Proof. Let $\tilde{U} = \{ U_\alpha : \alpha \in \Delta \}$ be a $P-D-$cover of $X$. Since $X$ is $P-D-$metaLindelöf, $\tilde{U}$ has a point countable parallel refinement $\tilde{V} = \{ V_\alpha \}_{i=1}^\infty$, which is also a $P-D-$cover of $(X, \tau_1, \tau_2)$. Since $X$ is countably $P-D-$metacompact, $\tilde{V}$ has a point finite parallel refinement $\tilde{W}$ of $\tilde{U}$.

Hence $(X, \tau_1, \tau_2)$ is $P-D-$metacompact.

**Corollary 2.8.** Every $P-D-$metalindelöf, countably $P-D-$metacompact space $(X, \tau_1, \tau_2)$ is $P-$metacompact space.
Corollary 2.9. Every $P$-metalindelöf, countably $P-D$-metacompact space $(X, \tau_1, \tau_2)$ is $P$-metacompact space.

Corollary 2.10. Every $P$-metalindelöf, countably $P$-metacompact space $(X, \tau_1, \tau_2)$ is $P$-metacompact space.

The last corollary can be also found in [11].

The following theorem is easily proved.

Theorem 2.10. Every $D$-metaLindelöf, countably $D$-metacompact space $(X, \tau_1, \tau_2)$ is $D$-metacompact space.

Theorem 2.11. Every $P - D$-compact space $(X, \tau_1, \tau_2)$ is $P$-compact.

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be a $P$-open cover of $(X, \tau_1, \tau_2)$. Then $\tilde{U}$ is a $P - D$-cover, and so it has a finite subcover. Hence the result. ∎

The converse of above theorem is not true as we will see in the following example.

Example 2.4. Let $X = \mathbb{R}$, $\tau_1 = \{\phi, X, \{1\}, \{1, 2\}\}$, $\tau_2 = \tau_{\text{cof}}$. Then $(\mathbb{R}, \tau_1, \tau_2)$ is $P$-compact but not $P - D$-metacompact, for the $P - D$-cover $\{\{x\} : x \in \mathbb{R}\}$ of $\mathbb{R}$ has no finite subcover.

Example 2.5. Let $X = \mathbb{R}$, $\beta_1 = \{X\} \cup \{x\} : x \in X - \{0\}$, $\beta_2 = \{X\} \cup \{x\} : x \in X - \{1\}$. Let $\tau_1$ and $\tau_2$ be the topologies on $X$ which are generated by the bases $\beta_1$ and $\beta_2$, respectively. Then $(\mathbb{R}, \tau_1, \tau_2)$ is $P-D$-metacompact and countably $P-D$-metacompact. On the other hand, $(\mathbb{R}, \tau_1, \tau_2)$ is not $P$-Lindelöf, since the $P$-open cover $\{\{x\} : x \in X\}$ of $X$ has no countable subcover. It is clear that $(\mathbb{R}, \tau_1, \tau_2)$ is not $P$-compact, since the $P$-open cover $\{\{x\} : x \in X\}$ of $X$ has no finite subcover, it is not compact space. So $(\mathbb{R}, \tau_1, \tau_2)$ is not $P - D$-Lindelöf nor $P - D$-compact.
The following definitions can be found in [10] and [6].

**Example 2.6.** (1) The bitopological space \((\mathbb{R}, \tau_{\text{dis}}, \tau_{\text{coc}})\) is \(P\)-metacompact, not \(B\)-metacompact, since the \(\tau_1\)-open cover \(\{\{x\} : x \in \mathbb{R}\}\) of \(\mathbb{R}\) has no point finite \(\tau_2\)-open refinement.

Also it is \(P - D\)-metacompact and \(B - D\)-metacompact. It is also countably \(P\)-metacompact. It is clear that \((\mathbb{R}, \tau_{\text{dis}}, \tau_{\text{coc}})\) is not \(P - D\)-Lindelöf space which is neither \(P\)-countably compact nor \(P\)-compact. So \((\mathbb{R}, \tau_{\text{dis}}, \tau_{\text{coc}})\) is \(P\)-Lindelöf space which is neither \(P - D\)-countably compact nor \(P - D\)-compact.

(2) Let \(\tau_s\) denotes the Sorgenfrey line topology on \(\mathbb{R}\). Then the bitopological space \((\mathbb{R}, \tau_s, \tau_u)\) is \(S - D\)-Lindelöf, so it is \(P - D\)-Lindelöf and \(D\)-Lindelöf. Also it is \(P - D\)-separable, so \((\mathbb{R}, \tau_s, \tau_u)\) is \(P - D\)-metacompact. It is clear that \((\mathbb{R}, \tau_s, \tau_u)\) is not \(B\)-metacompact, since the \(\tau_u\) open cover \(\{(-n, n) : n \in \mathbb{N}\}\) of \(\mathbb{R}\) has no point finite \(\tau_s\)-open refinement, because \(\tau_s \subsetneq \tau_u\). It is also clear that \((\mathbb{R}, \tau_s, \tau_u)\) is neither \(S - D\)-compact nor \(S - D\)-countably compact.

Remark 1. It is clear that every \(P - D\)-paracompact (\(S - D\)-paracompact) space is \(P - D\)-metacompact (\(S - D\)-metacompact).

**Theorem 2.12.** If the bitopological space \((X, \tau_1, \tau_2)\) is \(P - D\)-metacompact, then each \(\tau_1\)-closed subset of \(X\) is \(\tau_2\)-\(D\)-metacompact relative to \(X\), and each \(\tau_2\)-closed subset of \(X\) is \(\tau_1\)-\(D\)-metacompact relative to \(X\).

*Proof.* Let \(K \neq \emptyset\) be a \(\tau_1\)-closed subset of \(X\) and \(\bar{U} = \{U_\alpha : \alpha \in \Delta\}\) be a \(\tau_2\)-\(D\)-cover of \(K\). Then \(\hat{O} = \{X - K\} \cup \{U_\alpha : \alpha \in \Delta\}\) is a \(P - D\)-cover of \(X\). Since \((X, \tau_1, \tau_2)\) is \(P - D\)-metacompact, \(\hat{O}\) has a pairwise point finite parallel refinement, say \(\{V_\beta : \beta \in \Gamma\} \cup \{U^*_\alpha : \alpha \in \Delta\}\), where \(V_\beta\) is a \(\tau_1\)-\(D\)-sets for each \(\beta \in \Gamma\), and \(U^*_\alpha\) is a \(\tau_2\)-\(D\)-for each \(\alpha \in \Delta\). Thus \(\{U^*_\alpha : \alpha \in \Delta\}\) is a point finite parallel refinement of \(\hat{U}\). Hence \(K\) is a \(\tau_2\)-\(D\)-metacompact relative to \(X\). The proof of other case is similar. \(\square\)
Corollary 2.11. If the bitopological space $(X, \tau_1, \tau_2)$ is $P$-metacompact, then each $\tau_1$-closed subset of $X$ is $\tau_2-D$-metacompact relative to $X$, and each $\tau_2$-closed subset of $X$ is $\tau_1-D$-metacompact relative to $X$.

3. Product of D-metacompact Bitopological Spaces

Definition 3.1. A function $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is called
$P$-continuous ($P$-open, $P$-closed, $P$-homeomorphism, respectively) if the functions $f : (X, \tau_1) \longrightarrow (Y, \sigma_1)$ and $f : (X, \tau_2) \longrightarrow (Y, \sigma_2)$ are continuous (open, closed, homeomorphism, respectively).

Definition 3.2. A function $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is called $P$-perfect, if the function $f$ is $P$-continuous, $P$-closed, and for all $y \in Y$, the set $f^{-1}(y)$ is $P$-compact.

Theorem 3.1. Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a $P$-perfect function. If $X$ is locally indiscrete space, then $X$ is $P-D$-metacompact space if $Y$ is so.

Proof. Let $	ilde{U} = \{U_\alpha : \alpha \in \Delta\} \cup \{V_\beta : \beta \in \Gamma\}$ be any $P-D$-cover of $X$, where $\{U_\alpha : \alpha \in \Delta\}$ is a set of $\tau_1-D$-members of $\tilde{U}$ and $\{V_\beta : \beta \in \Gamma\}$ is a set of $\tau_2-D$-members of $\tilde{U}$.

Now, since $f$ is $P$-perfect, for every $y \in Y$ we have $f^{-1}(y)$ is $P$-compact subset of $X$. So there exist finite subsets $\Delta_1$ and $\Delta_2$ of $\Delta$ and $\Gamma$ respectively such that $f^{-1}(y) \subseteq \{\bigcup U_\alpha : \alpha \in \Delta_1\} \cup \{\bigcup V_\beta : \beta \in \Delta_2\}$. Now,

$O_{y_1} = Y - f(X - \bigcup U_\alpha : \alpha \in \Delta_1)$ is a $\tau_1$-open subset of $Y$ and $f^{-1}(O_{y_1}) \subseteq \{\bigcup U_\alpha : \alpha \in \Delta_1\}$.

$O_{y_2} = Y - f(X - \bigcup V_\beta : \beta \in \Delta_2)$ is a $\tau_2$-open subset of $Y$ and $f^{-1}(O_{y_2}) \subseteq \{\bigcup V_\beta : \beta \in \Delta_2\}$.

$y \in O_{y_1} \cup O_{y_2}$. So, $\tilde{O} = \{O_{y_1} : y \in Y\} \cup \{O_{y_2} : y \in Y\}$ is a $P$-open cover of $Y$. Since $Y$ is $P-D$-metacompact, $\tilde{O}$ has a pairwise point finite parallel refinement.
\[ O^* = \{ O^*_y : y \in Y \} \cup \{ O^*_y : y \in Y \}. \]

Now, \( O^*_y \) is a \( \tau_1 - D \)-open subset of \( X \) and \( O^*_y \) is a \( \tau_2 - D \)-open subset of \( X \).

Since \( f \) is perfect, the set \( \{ f^{-1}(O^*_y) : y \in Y \} \cup \{ f^{-1}(O^*_y) : y \in Y \} \) is a pairwise point finite parallel refinement of \( X \). So, \( X \) is \( P - D \)-metacompact. \( \square \)

**Corollary 3.1.** Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a \( P \)-perfect function. Then \( X \) is \( P \)-metacompact space if \( Y \) is so.

The last corollary can be also found in [11].

**Definition 3.3.** Let \( (X, \tau_1, \tau_2) \) and \( (Y, \sigma_1, \sigma_2) \) be bitopological spaces. Then the Cartesian product of \( (X, \tau_1, \tau_2) \) and \( (Y, \sigma_1, \sigma_2) \) is the bitopological space \( (X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2) \).

**Lemma 3.1.** If \( A \) is a compact subset of a topological space \( (X, \tau) \) and \( B \) is a compact subset of a topological space \( (Y, \sigma) \) and \( A \times B \subseteq W \); where \( W \) is an open subset of \( X \times Y \), then there exist open sets \( U \) and \( V \) in \( X \) and \( Y \) respectively such that \( A \times B \subseteq U \times V \subseteq W \).

**Theorem 3.2.** Let \( (X, \tau_1, \tau_2) \) and \( (Y, \sigma_1, \sigma_2) \) be bitopological spaces. If \( X \) is a Hausdorff compact, then the projection function \( P : X \times Y \rightarrow Y \) is \( P \)-closed.

**Proof.** To show that the projection function \( P : X \times Y \rightarrow Y \) is \( P \)-closed, we show that the projection functions \( P_1 : (X \times Y, \tau_1 \times \sigma_1) \rightarrow (Y, \sigma_1) \) and \( P_2 : (X \times Y, \tau_2 \times \sigma_2) \rightarrow (Y, \sigma_2) \) are closed. Let \( y \in Y \) and let \( U \) be an open set in \( (X \times Y, \tau_1 \times \sigma_1) \) such that \( P_1^{-1}(\{ y \}) \subseteq U \). So by (Wallace lemma), there exists a \( \sigma_1 \)-open set in \( Y \) say \( V_y \) such that \( P_1^{-1}(\{ y \}) = X \times \{ y \} \subseteq X \times V_y \subseteq U \). So, \( y \in V_y \) and \( P_1^{-1}(V_y) = X \times V_y \subseteq U \). So, \( P_1 : (X \times Y, \tau_1 \times \sigma_1) \rightarrow (Y, \sigma_1) \) is closed function. Similarly, we have \( P_2 : (X \times Y, \tau_2 \times \sigma_2) \rightarrow (Y, \sigma_2) \) is closed function. Hence, the projection function \( P : X \times Y \rightarrow Y \) is \( P \)-closed. \( \square \)
**Theorem 3.3.** Let \((X, \tau_1, \tau_2)\) and \((Y, \sigma_1, \sigma_2)\) be bitopological spaces such that \(X\) is a compact Hausdorff space and \(Y\) a \(P - D\)-metacompact space. Then \(X \times Y\) is a \(P - D\)-metacompact.

*Proof.* First, we know that the projection function \(P : X \times Y \rightarrow Y\) is \(P\)-continuous and for all \(y \in Y\), we have \(P^{-1}\{y\} = X \times \{y\} \simeq X\) is \(P - D\)-compact. Then \(P : X \times Y \rightarrow Y\) is \(P\)-perfect function. Since \(Y\) is \(P - D\)-metacompact, then \(X \times Y\) is \(P - D\)-metacompact by Theorem 3.2. \(\square\)

**Corollary 3.2.** The product of a compact Hausdorff bitopological space and a \(P\)-metacompact bitopological space is \(P\)-metacompact.

The last two corollaries can be also found in [11].

**Example 3.1.** The bitopological space \((\mathbb{R}, \tau_{cof}, \tau_{dis})\) is \(P\)-compact, so it is \(P\)-metacompact but not \(P - D\)-compact. Also it is \(B - D\)-metacompact, \(D\)-metacompact space since both \((\mathbb{R}, \tau_{cof})\) and \((\mathbb{R}, \tau_{dis})\) are \(D\)-metacompact.

The space \((\mathbb{R}^2, \tau_{cof} \times \tau_{cof}, \tau_{dis} \times \tau_{dis})\) is \(P - D\)-metacompact, but not \(P\)-compact nor \(P\)-Lindelöf, since the \(P\)-open cover

\[
\{\mathbb{R} \times (\mathbb{R} - \{0\})\} \cup \{(x, 0) : x \in \mathbb{R}\}
\]

for \(\mathbb{R}^2\) has no countable subcover. Hence the space \((\mathbb{R}^2, \tau_{cof} \times \tau_{cof}, \tau_{dis} \times \tau_{dis})\) is not \(P - D\)-compact nor \(P - D\)-Lindelöf.

**Lemma 3.2.** Let \(f : (X, \tau) \rightarrow (Y, \sigma)\) be a continuous, onto function. If \(\tilde{A} = \{A_\alpha : \alpha \in \Delta\}\) is a point finite family subset of \(X\), then \(\{f(A_\alpha) : \alpha \in \Delta\}\) is a point finite family subset of \(Y\).

**Theorem 3.4.** Let \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) be a \(P\)-continuous, \(P\)-closed, onto function and \(Y\) is locally indiscrete space. Then \(Y\) is \(P - D\)-metacompact, if \(X\) is so.
Proof. Let $\tilde{V} = \{U_\alpha : \alpha \in \Delta\} \cup \{V_\beta : \beta \in \Gamma\}$ be any $P-D-$cover of $Y$, where $\{U_\alpha : \alpha \in \Delta\}$ are $\sigma_1-D-$members of $\tilde{V}$ and $\{V_\beta : \beta \in \Gamma\}$ are $\sigma_2-D-$members of $\tilde{V}$. Since $f$ is $P-$continuous, onto function, the set

$$\tilde{U} = \{f^{-1}(U_\alpha) : \alpha \in \Delta\} \cup \{f^{-1}(V_\beta) : \beta \in \Gamma\}$$

is a $P-$open cover of $X$.

Since $X$ is $P-D-$metacompact space, there exists a pairwise point finite open parallel refinement of $\tilde{U}$, say $\tilde{U}^* = \{f^{-1}(U^*_\alpha) : \alpha \in \Delta\} \cup \{f^{-1}(V^*_\beta) : \beta \in \Gamma\}$. Thus, $\tilde{V}^* = \{U^*_\alpha : \alpha \in \Delta\} \cup \{V^*_\beta : \beta \in \Gamma\}$ is a pairwise point finite parallel refinement of $\tilde{V}$. So, $Y$ is $P-D-$metacompact. \qed

Corollary 3.3. Let $f : (X,\tau_1,\tau_2) \rightarrow (Y,\sigma_1,\sigma_2)$ be a $P-$continuous, $P-$closed, onto function. Then $Y$ is $P-D-$metacompact, if $X$ is $P-$metacompact.

Corollary 3.4. Let $f : (X,\tau_1,\tau_2) \rightarrow (Y,\sigma_1,\sigma_2)$ be a $P-$continuous, $P-$closed, onto function. Then $Y$ is $P-$metacompact, if $X$ is so.

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