ON ALMOST CONTRA $e^*\theta$-CONTINUOUS FUNCTIONS

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Abstract. The aim of this paper is to introduce and investigate some of fundamental properties of almost contra $e^*\theta$-continuous functions via $e^*\theta$-closed sets which are defined by Farhan and Yang [15]. Also, we obtain several characterizations of almost contra $e^*\theta$-continuous functions. Furthermore, we investigate the relationships between almost contra $e^*\theta$-continuous functions and separation axioms and $e^*\theta$-closedness of graphs of functions.

1. Introduction

In 2006, the concept of almost contra continuity [4], which is stronger than almost contra precontinuity [8] is introduced by Ekici and almost contra $\beta$-continuity [4] introduced by Baker, is defined. In 2017, some properties and characterizations of the notion of almost contra $\beta\theta$-continuous function [5] defined by Caldas via $\beta\theta$-closed sets are obtained. The notion of almost contra $e^*\theta$-continuity is stronger than almost contra $e^*$-continuity which is defined by us in this manuscript. In this paper, we introduce some new forms of contra $e^*$-continuity [9] defined by Ekici. Also, we obtain some characterizations of almost contra $e^*\theta$-continuous functions and investigate their some fundamental properties. Moreover, we investigate the relationships between almost contra $e^*\theta$-continuity and other related generalized forms of contra continuity.

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2. Preliminaries

Throughout this present paper, $X$ and $Y$ represent topological spaces. For a subset $A$ of a space $X$, $cl(A)$ and $int(A)$ denote the closure of $A$ and the interior of $A$, respectively. The family of all closed (resp. open) sets of $X$ is denoted by $C(X)$ (resp. $O(X)$). A subset $A$ is said to be regular open [28] (resp. regular closed [28]) if $A = int(cl(A))$ (resp. $A = cl(int(A))$). A point $x \in X$ is said to be $\delta$-cluster point [30] of $A$ if $int(cl(U)) \cap A \neq \emptyset$ for each open neighbourhood $U$ of $x$. The set of all $\delta$-cluster points of $A$ is called the $\delta$-closure [30] of $A$ and is denoted by $cl_{\delta}(A)$. If $A = cl_{\delta}(A)$, then $A$ is called $\delta$-closed [30], and the complement of a $\delta$-closed set is called $\delta$-open [30]. The set $\{x|(\exists U \in \tau)(x \in U)(int(cl(U)) \subseteq A)\}$ is called the $\delta$-interior of $A$ and is denoted by $int_{\delta}(A)$.

A subset $A$ is called $\alpha$-open [19] (resp. semiopen [17], $\delta$-semiopen [23], preopen [18], $\delta$-preopen [24], $b$-open [1], $e$-open [11], $e^*$-open [12], $a$-open [10]) if $A \subseteq int(cl(int(A)))$ (resp. $A \subseteq cl(int(A))$, $A \subseteq cl(int_{\delta}(A))$, $A \subseteq int(cl(A))$, $A \subseteq cl(int(A)) \cup int(cl(A))$, $A \subseteq cl(int_{\delta}(A)) \cup int(cl_{\delta}(A))$, $A \subseteq cl(int(cl_{\delta}(A))))$. The complement of an $\alpha$-open (resp. semiopen, $\delta$-semiopen, preopen, $\delta$-preopen, $b$-open, $e$-open, $e^*$-open, $a$-open) set is called $\alpha$-closed [19] (resp. semiclosed [17], $\delta$-semiclosed [23], preclosed [18], $\delta$-preclosed [24], $b$-closed [1], $e$-closed [11], $e^*$-closed [12], $a$-closed [10]). The intersection of all $e^*$-closed (resp. $a$-closed, semiclosed, $\delta$-semiclosed, preclosed, $\delta$-preclosed) sets of $X$ containing $A$ is called the $e^*$-closure [12] (resp. $a$-closure [10], semiclosure [17], $\delta$-semiclosure [23], preclosure [18], $\delta$-preclosure [24]) of $A$ and is denoted by $e^*cl(A)$ (resp. $a-cl(A)$, $scl(A)$, $\delta-scl(A)$, $pcl(A)$, $\delta-pcl(A)$). The union of all $e^*$-open (resp. $a$-open, semiopen, $\delta$-semiopen, preopen, $\delta$-preopen) sets of $X$ contained in $A$ is called the $e^*$-interior [12] (resp. $a$-interior [10], semiinterior [17], $\delta$-semiinterior [23], preinterior [18], $\delta$-preinterior [24]) of $A$ and is denoted by $e^*-int(A)$ (resp. $a-int(A)$, $sint(A)$, $\delta-sint(A)$, $pint(A)$, $\delta-pint(A)$).
A point $x$ of $X$ is called a $\theta$-cluster [30] point of $A$ if $cl(U) \cap A \neq \emptyset$ for every open set $U$ of $X$ containing $x$. The set of all $\theta$-cluster points of $A$ is called the $\theta$-closure [30] of $A$ and is denoted by $cl_\theta(A)$. A subset $A$ is said to be $\theta$-closed [30] if $A = cl_\theta(A)$. The complement of a $\theta$-closed set is called a $\theta$-open [30] set. A point $x$ of $X$ said to be a $\theta$-interior [30] point of a subset $A$, denoted by $int_\theta(A)$, if there exists an open set $U$ of $X$ containing $x$ such that $cl(U) \subseteq A$.

A point $x \in X$ is said to be a $\theta$-semicluster point [16] of a subset $S$ of $X$ if $cl(U) \cap A \neq \emptyset$ for every semiopen $U$ containing $x$. The set of all $\theta$-semicluster points of $A$ is called the $\theta$-semiclosure of $A$ and is denoted by $\theta scl(A)$. A subset $A$ is called $\theta$-semiclosed [16] if $A = \theta scl(A)$. The complement of a $\theta$-semiclosed set is called $\theta$-semiopen.

The union of all $e^*$-open sets of $X$ contained in $A$ is called the $e^*$-interior [12] of $A$ and is denoted by $e^*int(A)$. A subset $A$ is said to be $e^*$-regular [15] if it is $e^*$-open and $e^*$-closed. The family of all $e^*$-regular subsets of $X$ is denoted by $e^*R(X)$.

A point $x$ of $X$ is called an $e^*$-$\theta$-cluster point of $A$ if $e^*cl(U) \cap A \neq \emptyset$ for every $e^*$-open set $U$ containing $x$. The set of all $e^*$-$\theta$-cluster points of $A$ is called the $e^*$-$\theta$-closure [15] of $A$ and is denoted by $e^*cl_\theta(A)$. A subset $A$ is said to be $e^*$-$\theta$-closed if $A = e^*cl_\theta(A)$. The complement of an $e^*$-$\theta$-closed set is called an $e^*$-$\theta$-open [15] set. A point $x$ of $X$ said to be an $e^*$-$\theta$-interior [15] point of a subset $A$, denoted by $e^*int_\theta(A)$, if there exists an $e^*$-open set $U$ of $X$ containing $x$ such that $e^*cl(U) \subseteq A$.

Also it is noted in [15] that

$$e^*\text{regular} \Rightarrow e^*\theta\text{open} \Rightarrow e^*\text{open}.$$ 

The family of all $e^*$-$\theta$-open (resp. $e^*$-$\theta$-closed, $e^*$-open, $e^*$-closed, regular open, regular closed, $\delta$-open, $\delta$-closed, $\theta$-open, $\theta$-closed, $\theta$-semiopen, $\theta$-semiclosed, semiopen, semiclosed, preopen, preclosed, $\delta$-semiopen, $\delta$-semiclosed, $\delta$-preopen, $\delta$-preclosed, $\alpha$-open, $\alpha$-closed) subsets of $X$ is denoted by $e^*\theta O(X)$ (resp. $e^*\theta C(X)$, $e^*O(X)$, $e^*C(X)$, $RO(X)$, $RC(X)$, $\delta O(X)$, $\delta C(X)$, $\theta O(X)$, $\theta C(X)$, $\theta SO(X)$, $\theta SC(X)$, $SO(X)$, $SC(X)$, ...
The family of all open (resp. closed, $e^*\theta$-open, $e^*\theta$-closed, $e^*$-open, $e^*$-closed, regular open, regular closed, $\delta$-open, $\delta$-closed, $\theta$-open, $\theta$-closed, $\theta$-semiopen, $\theta$-semiclosed, semiopen, semiclosed, preopen, preclosed, $\delta$-semiopen, $\delta$-semiclosed, $\delta$-preopen, $\delta$-preclosed, $a$-open, $a$-closed) sets of $X$ containing a point $x$ of $X$ is denoted by $O(X,x)$ (resp. $C(X,x)$, $e^*\theta O(X,x)$, $e^*\theta C(X,x)$, $e^*O(X,x)$, $e^*C(X,x)$, $RO(X,x)$, $RC(X,x)$, $\delta O(X,x)$, $\delta C(X,x)$, $\theta O(X,x)$, $\theta C(X,x)$, $\theta SO(X,x)$, $\theta SC(X,x)$, $SO(X,x)$, $SC(X,x)$, $PO(X,x)$, $PC(X,x)$, $\delta SO(X,x)$, $\delta SC(X,x)$, $\delta PO(X,x)$, $\delta PC(X,x)$, $aO(X,x)$, $aC(X,x)$).

We shall use the well-known accepted language almost in the whole of the proofs of the theorems in this article.

Lemma 2.1. [12] Let $A$ be a subset of a space $X$, then the followings hold:

1. $e^*\text{-cl}(X \setminus A) = X \setminus e^*\text{-int}(A)$,
2. $x \in e^*\text{-cl}(A)$ if and only if $A \cap U \neq \emptyset$ for every $U \in e^*O(X,x)$,
3. $A$ is $e^*C(X)$ if and only if $A = e^*\text{-cl}(A)$,
4. $e^*\text{-int}(A) \subseteq e^*C(X)$,
5. $e^*\text{-int}(A) = A \cap cl(int(cl_\delta(A)))$.

Lemma 2.2. [10, 23, 24] Let $A$ be a subset of a space $X$, then the followings hold:

1. $a\text{-cl}(A) = A \cup cl(int(cl_\delta(A)))$,
2. $\delta\text{-scl}(A) = A \cup int(cl_\delta(A))$,
3. $\delta\text{-pcl}(A) = A \cup cl(int_\delta(A))$.

Lemma 2.3. [15] The following properties hold for the $e^*\theta$-closure of a subset $A$ of a topological space $X$.

1. $A \subseteq e^*\text{-cl}(A) \subseteq e^*\text{-cl}_\theta(A)$,
2. If $A \in e^*\theta O(X)$, then $e^*\text{-cl}_\theta(A) = e^*\text{-cl}(A)$,
3. If $A \subseteq B$, then $e^*\text{-cl}_\theta(A) \subseteq e^*\text{-cl}_\theta(B)$,
Lemma 2.4. [15] Let $A$ be a subset of a topological space $X$, then the followings hold:

1. If $A \in e^*O(X)$, then $e^* \text{cl}_\theta(A) \subseteq e^*R(X)$,
2. $A \subseteq e^*R(X)$ if and only if $A \in e^*O(X) \cap e^*C(X)$,
3. $A$ is $e^*\theta$-open in $X$ if and only if for each $x \in A$ there exists $U \in e^*R(X, x)$ such that $x \in U \subseteq A$.

Definition 2.1. Let $A$ be a subset of a space $X$. The intersection of all regular open sets in $X$ containing $A$ is called the $r$-kernel of $A$ [9] and is denoted by $rker(A)$.

Lemma 2.5. [9] The following properties hold for subsets $A$ and $B$ of a space $X$.

1. $x \in rker(A)$ if and only if $A \cap F \neq \emptyset$ for any $F \in RC(X, x)$,
2. $A \subseteq rker(A)$,
3. If $A$ is regular open in $X$, then $A = rker(A)$,
4. If $A \subseteq B$, then $rker(A) \subseteq rker(B)$.


1. $\text{cl}(\text{int}_\delta(A)) = \text{cl}_\delta(\text{int}_\delta(A))$,
2. $\text{int}(\text{cl}_\delta(A)) = \text{int}_\delta(\text{cl}_\delta(A))$.

Lemma 2.7. Let $A$ be a subset of a topological space $X$. If $A$ is an $e^*$-open set in $X$, then $\text{int}_\delta(X \setminus A) = X \setminus \text{cl}_\delta(A) \in RO(X)$.
Proof. Let $A \in e^*O(X)$.

$A \in e^*O(X) \Rightarrow A \subseteq cl(int(cl_\delta(A)))$

$\Rightarrow cl_\delta(A) \subseteq cl(cl(int(cl_\delta(A))))$ \hspace{1cm} \text{Lemma 2.6}

$\Rightarrow cl_\delta(A) \subseteq cl(cl(int(cl_\delta(A)))) = cl_\delta(int(cl_\delta(A)))$

$\Rightarrow cl_\delta(A) \subseteq cl(cl(int(cl_\delta(A))))$ \hspace{1cm} \text{Lemma 2.6}

$\Rightarrow cl(int(cl_\delta(A))) = int(cl(cl_\delta(A))) \subseteq cl_\delta(A) \ldots (\ast)$

$int(cl_\delta(A)) \subseteq cl_\delta(A) \Rightarrow cl(int(cl_\delta(A))) = cl_\delta(int(cl_\delta(A))) \subseteq cl_\delta(cl_\delta(A)) = cl_\delta(A)$

$\Rightarrow \{cl(int(cl_\delta(A))) = int(cl(cl_\delta(A)))\} \ldots (\ast\ast)$

$(\ast), (\ast\ast) \Rightarrow \{cl_\delta(A) = int(cl(cl_\delta(A))) \Rightarrow cl_\delta(A) \in RO(X).$ \hfill \Box

**Definition 2.2.** A function $f : X \rightarrow Y$ is said to be:

a) $e^*\theta$-continuous (briefly $e^*\theta.c.$) if $f^{-1}[V]$ is $e^*-\theta$-closed in $X$ for every $V \in C(Y)$,

b) almost $e^*\theta$-continuous (briefly a.$e^*\theta.c.$) if $f^{-1}[V]$ is $e^*-\theta$-closed in $X$ for every regular closed set $V$ in $Y$,

c) contra $R$-map [9] (resp. contra continuous [7], contra $e^*\theta$-continuous [3], contra $e^*$-continuous [13]) if $f^{-1}[V]$ is regular closed (resp. closed, $e^*\theta$-closed, $e^*$-closed) in $X$ for every regular open (resp. open, open, open) set $V$ in $Y$,

d) almost contra precontinuous [8] (resp. almost contra continuous [4], almost contra $\beta$-continuous [4], almost contra $e^*$-continuous) if $f^{-1}[V]$ is preclosed (resp. closed, $\beta$-closed, $e^*$-closed) in $X$ for every regular open set $V$ in $Y$.

**Lemma 2.8.** [25] For a topological space $(X, \tau)$ the followings are equivalent:

(1) $(X, \tau)$ is almost regular;

(2) For each point $x \in X$ and each neighbourhood $M$ of $x$, there exists a regular open neighbourhood $V$ of $x$ such that $cl(V) \subseteq int(cl(M))$. 

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3. Almost Contra $e^*\theta$-Continuous Functions

**Definition 3.1.** A function $f : X \to Y$ is said to be almost contra $e^*\theta$-continuous (briefly a.c.$e^*\theta$.c.) if $f^{-1}[V]$ is $e^*-\theta$-closed in $X$ for each regular open set $V$ of $Y$.

**Theorem 3.1.** For a function $f : X \to Y$, the following properties are equivalent:

1. $f$ is almost contra $e^*\theta$-continuous;
2. The inverse image of each regular closed set in $Y$ is $e^*\theta$-open in $X$;
3. For each point $x \in X$ and each $V \in RC(Y, f(x))$, there exists $U \in e^\theta O(X, x)$ such that $f[U] \subseteq V$;
4. For each point $x \in X$ and each $V \in SO(Y, f(x))$, there exists $U \in e^\theta O(X, x)$ such that $f[U] \subseteq \text{cl}(V)$;
5. $f^{-1} [e^*\text{-cl}(A)] \subseteq \text{rker}(f[A])$ for every subset $A$ of $X$;
6. $e^*-\text{cl}(f^{-1}[B]) \subseteq f^{-1}[\text{rker}(B)]$ for every subset $B$ of $Y$;
7. $f^{-1} [\text{cl}_\delta(V)]$ is $e^*\theta$-open for every $V \in e\theta O(Y)$;
8. $f^{-1} [\text{cl}_\delta(V)]$ is $e^*\theta$-open for every $V \in \delta SO(Y)$;
9. $f^{-1} [\text{int}(\text{cl}_\delta(V))]$ is $e^*\theta$-closed for every $V \in \delta PO(Y)$;
10. $f^{-1} [\text{int}(\text{cl}_\delta(V))]$ is $e^*\theta$-closed for every $V \in O(Y)$;
11. $f^{-1} [\text{cl}(\text{int}_\delta(V))]$ is $e^*\theta$-open for every $V \in C(Y)$.

**Proof.**

1. $\Rightarrow$ 2: Let $V \in RC(Y)$.
   
   \[ V \in RC(Y) \Rightarrow \forall V \in RO(Y) \implies f^{-1} [V] = f^{-1} [\text{cl}(V)] \in e^\theta C(X) \]

   \[ \Rightarrow f^{-1} [V] \in e^\theta O(X). \]

2. $\Rightarrow$ 3: Let $x \in X$ and $V \in RC(Y, f(x))$.
   
   \[ (x \in X)(V \in RC(Y, f(x))) \implies U := f^{-1} [V] \in e^\theta O(X, x)(f[U] \subseteq V). \]

3. $\Rightarrow$ 4: Let $x \in X$ and $V \in SO(Y, f(x))$. 

\[ V \in SO(Y, f(x)) \Rightarrow cl(int(V)) \subseteq RC(Y, f(x)) \]

\[
\Rightarrow (\exists U \in e^*O(X, x))(f[U] \subseteq cl(int(V)) \subseteq cl(V)).
\]

(4) \Rightarrow (5) : Let \( A \subseteq X \) and \( x \notin f^{-1}[rker(f[A])]. \)

\[ x \notin f^{-1}[rker(f[A])] \Rightarrow f(x) \notin rker(f[A]) \Rightarrow (\exists F \in RC(Y, f(x)))(F \cap f[A] = \emptyset) \]

\[
\Rightarrow (\exists F \in SO(Y, f(x)))(f^{-1}[F] \cap A = \emptyset)
\]

\[
\Rightarrow (\exists U \in e^*O(X, x))(f[U] \subseteq cl(F) = F)\left(f^{-1}[F] \cap A = \emptyset\right)
\]

\[
\Rightarrow (\exists U \in e^*O(X, x)) \subseteq f^{-1}[F] \cap A = \emptyset
\]

\[
\Rightarrow (\exists U \in e^*O(X, x))(U \cap A = \emptyset)
\]

\[
x \notin e^*-cl_\theta(A).
\]

(5) \Rightarrow (6) : Let \( B \subseteq Y. \)

\[
B \subseteq Y \Rightarrow f^{-1}[B] \subseteq X
\]

\[
\Rightarrow f[e^*-cl_\theta(f^{-1}[B])] \subseteq rker(f[f^{-1}[B]]) \subseteq rker(B)
\]

\[
\Rightarrow e^*-cl_\theta(f^{-1}[B]) \subseteq f^{-1}[rker(B)].
\]

(6) \Rightarrow (7) : Let \( V \in e^*O(Y). \)

\[
V \in e^*O(Y) \text{Lemma 2.7} \ \\ \ \ \ \ cl_\delta(V) \in RO(Y)
\]

\[
\Rightarrow e^*-cl_\theta(f^{-1}[\delta cl_\delta(V)]) \subseteq f^{-1}[rker(\delta cl_\delta(V))] = f^{-1}[\delta cl_\delta(V)]
\]

\[
\Rightarrow e^*-int_\theta(f^{-1}[\delta cl_\delta(V)]) \subseteq f^{-1}[\delta cl_\delta(V)]
\]

\[
\Rightarrow f^{-1}[\delta cl_\delta(V)] \subseteq e^*-int_\theta(f^{-1}[\delta cl_\delta(V)])
\]

\[
\Rightarrow f^{-1}[\delta cl_\delta(V)] \in e^*O(X).
\]

(7) \Rightarrow (8) : This is obvious since every \( \delta \)-semiopen set is \( e^* \)-open.

(8) \Rightarrow (9) : Let \( V \in \delta PO(Y). \)

\[
V \in \delta PO(Y) \Rightarrow int_\delta(V) \in \delta SO(Y)
\]

\[
\Rightarrow f^{-1}[\delta cl_\delta(int_\delta(V))] \in e^*O(X)
\]
\( \Rightarrow f^{-1} [\text{int}_\delta (cl_\delta (V))] \in e^*O(X) \)
\( \Rightarrow f^{-1} [\text{int} (cl_\delta (V))] \in e^*C(X). \)

(9) \( \Rightarrow \) (10) : This is obvious since every open set is \( \delta \)-preopen.

(10) \( \Rightarrow \) (11) : Clear.

(11) \( \Rightarrow \) (1) : Let \( V \in RO(Y). \)
\( V \in RO(Y) \Rightarrow (V = \text{int}(cl_\delta (V))) (\forall V \in C(Y)) \)

(11) \( \Rightarrow \)
\( \Rightarrow f^{-1}[\forall V = f^{-1}[\forall V = f^{-1} [\text{int}_\delta (cl_\delta (V))] = f^{-1} [\text{int}_\delta (\forall V)]]) \in e^*O(X) \)
\( \Rightarrow f^{-1}[\forall V] \in e^*C(X). \)
\( \square \)

**Lemma 3.1.** For a subset \( A \) of a topological space \( X \), the following properties hold:
(1) If \( A \in e^*O(X) \), then \( a-cl(A) = cl_\delta (A) \),
(2) If \( A \in \delta SO(X) \), then \( \delta \)-pcl \( (A) = cl_\delta (A) \),
(3) If \( A \in \delta PO(X) \), then \( \delta \)-scl \( (A) = \text{int}(cl_\delta (A)) \),
(4) If \( A \in PO(X) \), then scl \( (A) = \text{int}(cl(A)) \).

**Proof.** (1) Let \( A \in e^*O(X) \).
\( A \in e^*O(X) \Rightarrow A \subseteq cl(\text{int}(cl_\delta (A))) \)
\( \Rightarrow cl_\delta (A) \subseteq cl_\delta (cl(\text{int}(cl_\delta (A)))) = cl(\text{int}(cl_\delta (A))) \)
\( \Rightarrow A \cup cl_\delta (A) = cl_\delta (A) \subseteq A \cup cl(\text{int}(cl_\delta (A))) = a-cl(A) \ldots (*) \)
\( \delta C(X) \subseteq aC(X) \Rightarrow a-cl(A) \subseteq cl_\delta (A) \ldots (***) \)

(1), (***) \( \Rightarrow a-cl(A) = cl_\delta (A) \).

(2) Let \( A \in \delta SO(X) \).
\( A \in \delta SO(X) \Rightarrow A \subseteq cl(\text{int}_\delta (A)) \quad \text{Lemma 2.6} \quad cl_\delta (\text{int}_\delta (A)) \)
\( \Rightarrow cl_\delta (A) \subseteq cl_\delta (cl_\delta (\text{int}_\delta (A))) = cl_\delta (\text{int}_\delta (A)) = cl(\text{int}_\delta (A)) \)
\( \delta \)-pcl \( (A) = A \cup cl(\text{int}_\delta (A)) \)
\( \delta C(X) \subseteq \delta PC(X) \Rightarrow \delta \)-pcl \( (A) \subseteq cl_\delta (A) \)
\( \Rightarrow \delta \)-pcl \( (A) = cl_\delta (A). \)
(3) Let $A \in \delta PO(X)$.
\[
A \in \delta PO(X) \Rightarrow A \subseteq int(\delta cl(A)) \quad \text{and} \quad \delta scl(A) = A \cup int(\delta cl(A)).
\]

(4) [20].

Corollary 3.1. For a function $f : X \to Y$, the following properties are equivalent:

1. $f$ is almost contra $e^*\theta$-continuous;
2. $f^{-1}[a-cl(A)]$ is $e^*\theta$-open for every $A \in e^O(Y);
3. $f^{-1}[\delta pcl(A)]$ is $e^*\theta$-open for every $A \in \delta SO(Y);
4. $f^{-1}[\delta scl(A)]$ is $e^*\theta$-closed for every $A \in \delta PO(Y)$.

Proof. It follows from Lemma 3.1.

Theorem 3.2. For a function $f : X \to Y$, the following properties are equivalent:

1. $f$ is almost contra $e^*\theta$-continuous;
2. $f^{-1}[V]$ is $e^*\theta$-open in $X$ for each $\theta$-semiopen set of $Y;
3. $f^{-1}[V]$ is $e^*\theta$-closed in $X$ for each $\theta$-semiclosed set of $Y;
4. $f^{-1}[V] \subseteq e^*-int_\theta(f^{-1}[cl(V)])$ for every $V \in SO(Y);
5. $f[e^*\text{cl}_\theta(A)] \subseteq \theta \text{ scl}(f[A])$ for every subset $A$ of $X$;
6. $e^*\text{cl}_\theta(f^{-1}[B]) \subseteq f^{-1}[\theta \text{ scl}(B)]$ for every subset $B$ of $Y$;
7. $e^*\text{cl}_\theta(f^{-1}[V]) \subseteq f^{-1}[\theta \text{ scl}(V)]$ for every open subset $V$ of $Y$;
8. $e^*\text{cl}_\theta(f^{-1}[V]) \subseteq f^{-1}[\text{ scl}(V)]$ for every open subset $V$ of $Y$;
9. $e^*\text{cl}_\theta(f^{-1}[V]) \subseteq f^{-1}[\text{ int}(cl(V))$ for every open subset $V$ of $Y$.

Proof. (1) $\Rightarrow$ (2) : Let $V \in \theta SO(Y)$.
\[
V \in \theta SO(Y) \Rightarrow (\exists A \subseteq RC(Y))(V = \cup A) \quad \text{(1)} \Rightarrow
\]
\[
\Rightarrow f^{-1}[V] = \cup \{f^{-1}[A] | A \in A\} \in e^*\theta O(X).
\]

(2) $\Rightarrow$ (3) : Obvious.

(3) $\Rightarrow$ (4) : Let $V \in SO(Y)$. 

\[
\]
\[ V \in SO(Y) \Rightarrow \text{cl}(V) \in \theta SC(Y) \quad (3) \]
\[ \Rightarrow f^{-1}[\text{cl}(V)] \in e^*\theta C(X) \Rightarrow f^{-1}[\text{cl}(V)] \in e^*\theta C(X) \]
\[ \Rightarrow f^{-1}[\text{cl}(V)] \in e^*\theta O(X) \Rightarrow f^{-1}[V] \subseteq f^{-1}[\text{cl}(V)] = e^*-\text{int}_{\theta}(f^{-1}[\text{cl}(V)]). \]

(4) \Rightarrow (5) : Let \( A \subseteq X \) and \( x \notin f^{-1}[\theta-scl(f[A])]. \)
\[ x \notin f^{-1}[\theta-scl(f[A])] \Rightarrow f(x) \notin \theta-scl(f[A]) \Rightarrow (\exists U \in SO(Y, f(x)))(cl(U) \cap f[A] = \emptyset) \]
\[ \Rightarrow (\exists U \in SO(Y, f(x)))(e^*-\text{int}_{\theta}(f^{-1}[cl(U)]) \cap A = \emptyset) \quad (4) \]
\[ V := e^*-\text{int}_{\theta}(f^{-1}[cl(U)]) \]
\[ \Rightarrow (\exists V \in e^*\theta O(X, x))(V \cap A = \emptyset) \]
\[ \Rightarrow x \notin e^*-\text{cl}_{\theta}(A). \]

(5) \Rightarrow (6) : Let \( B \subseteq Y. \)
\[ B \subseteq Y \Rightarrow f^{-1}[B] \subseteq X \quad (5) \]
\[ \Rightarrow f[e^*-\text{cl}_{\theta}(f^{-1}[B])] \subseteq \theta-scl(f[f^{-1}[B]]) \subseteq \theta-scl(B) \]
\[ \Rightarrow e^*-\text{cl}_{\theta}(f^{-1}[B]) \subseteq f^{-1}[\theta-scl(B)]. \]

(6) \Rightarrow (7) : Obvious.

(7) \Rightarrow (8) : This is obvious since \( \theta-scl(V) = scl(V) \) for an open set \( V. \)

(8) \Rightarrow (9) : Obvious from Lemma 3.1(4).

(9) \Rightarrow (1) : Let \( V \in RO(Y). \)
\[ V \in RO(Y) \subseteq O(Y) \quad (9) \]
\[ \Rightarrow e^*-\text{cl}_{\theta}(f^{-1}[V]) \subseteq f^{-1}[\text{in}(cl(V))] = f^{-1}[V] \]
\[ \Rightarrow f^{-1}[V] \in e^*\theta C(X). \]

We recall that a topological space \( X \) is said to be extremally disconnected if the closure of every open set of \( X \) is open in \( X. \)

**Lemma 3.2.** Let \( X \) be a topological space. If \( X \) is an extremally disconnected space, then \( RO(X) = RC(X). \)
Theorem 3.3. Let $f : X \to Y$ be a function. If $Y$ is extremally disconnected, then the following properties are equivalent:

(1) $f$ is almost contra $e^*\theta$-continuous;

(2) $f$ is almost $e^*\theta$-continuous.

Proof. The proof is obvious from Lemma 3.2. □

Remark 1. From Definitions 2.2 and 3.1, we have the following diagram:

\[
\begin{array}{cccc}
\text{contra } e^*\theta\text{-con.} & \to & \text{contra } e^*\text{-con.} & \to & \text{almost contra } e^*\text{-con.} \\
\downarrow & & \uparrow & & \uparrow \\
& \text{almost contra } e^*\theta\text{-con.} & \to & \text{almost contra } \beta\text{-con.} & \to & \text{almost contra pre-con.} \\
& \text{contra } R\text{-map} & \to & \text{almost contra con.} & \to & \text{almost contra pre-con.}
\end{array}
\]

Example 3.1. Let $X := \{a, b, c, d\}$ and $\tau := \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. It is not difficult to see $e^*\theta O(X) = e^* O(X) = 2^X \setminus \{\{c\}, \{d\}, \{c, d\}\}$. Then the identity function $f : (X, \tau) \to (X, \tau)$ is almost contra $e^*\theta$-continuous and so almost contra $e^*$-continuous but $f$ is neither contra $e^*\theta$-continuous nor contra $e^*$-continuous.

Example 3.2. Let $X := \{a, b, c, d\}$ and $\tau := \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}\}$. It is not difficult to see $e^*\theta O(X) = e^* O(X) = 2^X \setminus \{\{c\}, \{d\}, \{b, c\}, \{c, d\}, \{b, c, d\}\}$ and $\beta O(X) = 2^X \setminus \{\{c\}, \{d\}, \{b, c\}, \{c, d\}, \{b, c, d\}\}$. Define the function $f : (X, \tau) \to (X, \tau)$ by $f = \{(a, b), (b, a), (c, c), (d, d)\}$. Then $f$ is almost contra $e^*\theta$-continuous but it is not almost contra $\beta$-continuous.

Theorem 3.4. If $f : X \to Y$ is an almost contra $e^*\theta$-continuous function which satisfies the property $e^*\text{-}\text{int}_\theta(f^{-1}[\text{cl}_\delta(V)]) \subseteq f^{-1}[V]$ for each open set $V$ of $Y$, then $f$ is $e^*\theta$-continuous.
Proof. Let $V \in O(Y)$. 
\[
V \in O(Y) \quad \text{Theorem 3.1(7)} \quad f \text{ is a.c.} \, e^*\theta-c.
\]
\[
\Rightarrow f^{-1}[V] \subseteq f^{-1}[\text{cl}_e(V)] = e^*\text{-int}_\theta(f^{-1}[\text{cl}(V)]) \subseteq e^*\text{-int}_\theta(f^{-1}[V]) \subseteq f^{-1}[V]
\]
\[
\Rightarrow f^{-1}[V] = e^*\text{-int}_\theta(f^{-1}[V])
\]
\[
\Rightarrow f^{-1}[V] \in e^*\theta O(X).
\]

We recall that a topological space is said to be $P_\Sigma$ [29] if for any open set $V$ of $X$ and each $x \in V$, there exists a regular closed set $F$ of $X$ containing $x$ such that $x \in F \subseteq V$.

**Theorem 3.5.** If $f : X \to Y$ is an almost contra $e^*\theta$-continuous function and $Y$ is $P_\Sigma$, then $f$ is $e^*\theta$-continuous.

**Proof.** Let $V \in O(Y)$.

\[
y \in V \in O(Y) \quad \Rightarrow \quad (\exists F \in RC(Y, y))(F \subseteq V) \quad \Rightarrow \quad \left\{ \begin{array}{l}
A := \{ F | y \in V \Rightarrow (\exists F \in RC(Y, y))(F \subseteq V) \} \\
\cup A = V \\
f \text{ is a.c.} \, e^*\theta-c.
\end{array} \right\} \Rightarrow \Rightarrow \Rightarrow \\
\Rightarrow f^{-1}[V] = \bigcup_{F \in A} f^{-1}[F] \in e^*\theta O(X).
\]

**Definition 3.2.** A function $f : X \to Y$ is said to be:

a) $R$-map [6] if $f^{-1}[A]$ is regular closed in $X$ for every regular closed $A$ of $Y$,

b) weakly $e^*$-irresolute [22] if $f^{-1}[A]$ is $e^*\theta$-open in $X$ for every $e^*\theta$-open set $A$ of $Y$,

c) pre-$e^*\theta$-closed if $f[A]$ is $e^*\theta$-closed in $Y$ for every $e^*\theta$-closed $A$ of $X$.

**Theorem 3.6.** Let $f : X \to Y$ and $g : Y \to Z$ be two functions. Then the following properties hold:

1. If $f$ is almost contra $e^*\theta$-continuous and $g$ is an $R$-map, then $g \circ f : X \to Z$ is almost contra $e^*\theta$-continuous,

2. If $f$ is almost $e^*\theta$-continuous and $g$ is a contra $R$-map, then $g \circ f : X \to Z$ is
almost contra $e^\ast\theta$-continuous,

(3) If $f$ is weakly $e^\ast$-irresolute and $g$ is almost contra $e^\ast\theta$-continuous, then $g \circ f : X \to Z$ is almost contra $e^\ast\theta$-continuous.

Proof. Routine. □

Theorem 3.7. If $f : X \to Y$ is a pre-$e^\ast\theta$-closed surjection and $g : Y \to Z$ is a function such that $g \circ f : X \to Z$ is almost contra $e^\ast\theta$-continuous, then $g$ is almost contra $e^\ast\theta$-continuous.

Proof. Let $V \in RO(Z)$.

\[
V \in RO(Z) \quad \text{and} \quad g \circ f \text{ is a.c.}e^\ast\theta.c. \quad \Rightarrow \quad (g \circ f)^{-1}[V] = f^{-1}[g^{-1}[V]] \in e^\ast\theta C(X) \quad \Rightarrow \quad f \text{ is pre-$e^\ast\theta$-closed surjection}.
\]

\[
\Rightarrow f[f^{-1}[g^{-1}[V]]] = g^{-1}[V] \in e^\ast\theta C(Y). \quad \square
\]

Theorem 3.8. Let $\{X_\alpha | \alpha \in \Lambda\}$ be any family of topological spaces. If $f : X \to \Pi X_\alpha$ is an almost contra $e^\ast\theta$-continuous function, then $Pr_\alpha \circ f : X \to X_\alpha$ is almost contra $e^\ast\theta$-continuous for each $\alpha \in \Lambda$ where $Pr_\alpha$ is the projection of $\Pi X_\alpha$ onto $X_\alpha$.

Proof. Let $\alpha \in \Lambda$ and $U_\alpha \in RO(X_\alpha)$.

\[
\alpha \in \Lambda \Rightarrow Pr_\alpha \text{ is open and continuous} \Rightarrow Pr_\alpha \text{ is R-map} \quad \Rightarrow \quad U_\alpha \in RO(X_\alpha) \quad \Rightarrow \quad (Pr_\alpha \circ f)^{-1}[U_\alpha] = f^{-1}[Pr_\alpha^{-1}[U_\alpha]] \in e^\ast\theta C(X).
\]

Definition 3.3. A function $f : X \to Y$ is called weakly $e^\ast\theta$-continuous (briefly w.$e^\ast\theta$.c.) if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists a $U \in e^\ast\theta O(X, x)$ such that $f[U] \subseteq cl(V)$.

Theorem 3.9. Let $f : X \to Y$ be a function. Then the following properties hold:

(1) If $f$ is almost contra $e^\ast\theta$-continuous, then it is weakly $e^\ast\theta$-continuous,
(2) If $f$ is weakly $e^\ast \theta$-continuous and $Y$ is extremally disconnected, then $f$ is almost contra $e^\ast \theta$-continuous.

Proof. (1) Let $x \in X$ and $V \in O(Y, f(x))$.

$$(x \in X)(V \in O(Y, f(x))) \Rightarrow cl(V) \in RC(Y, f(x)) \Rightarrow cl(V) \subseteq O(Y, f(x)) \Rightarrow U := f^{-1}[cl(V)] \subseteq O(X, f(x)) \Rightarrow f$ is a.c.$e^\ast \theta.c.$

$$(V \in RC(Y))(x \in f^{-1}[V]) \Rightarrow (V \in RC(Y, f(x)))(cl(V) = V) \Rightarrow Y \text{ is extremally disconnected} \Rightarrow \exists U \in e^\ast \theta O(X, x)(f[U] \subseteq cl(V) = V) \Rightarrow f^{-1}[V] \in e^\ast \theta O(X).$$

4. Some Fundamental Properties

Definition 4.1. A topological space $X$ is said to be:

a) $e^\ast \theta$-$T_0$ if for any distinct pair of points $x$ and $y$ in $X$, there is an $e^\ast \theta$-open set $U$ in $X$ containing $x$ but not $y$ or an $e^\ast \theta$-open set $V$ in $X$ containing $y$ but not $x$,

b) $e^\ast \theta$-$T_1$ if for any distinct pair of points $x$ and $y$ in $X$, there is an $e^\ast \theta$-open set $U$ in $X$ containing $x$ but not $y$ and an $e^\ast \theta$-open set $V$ in $X$ containing $y$ but not $x$,

c) $e^\ast \theta$-$T_2$ (resp. $e^\ast$-$T_2$ [13, 14]) if for every pair of distinct points $x$ and $y$, there exist two $e^\ast \theta$-open (resp. $e^\ast$-open) sets $U$ and $V$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Theorem 4.1. For a topological space $X$, the following properties are equivalent:

(1) $(X, \tau)$ is $e^\ast \theta$-$T_0$;

(2) $(X, \tau)$ is $e^\ast \theta$-$T_1$;

(3) $(X, \tau)$ is $e^\ast \theta$-$T_2$;
(4) $(X, \tau)$ is $e^*-T_2$;

(5) For every pair of distinct points $x, y \in X$, there exist $U \in e^*O(X, x)$ and $V \in e^*O(X, y)$ such that $e^*\text{-cl}(U) \cap e^*\text{-cl}(V) = \emptyset$;

(6) For every pair of distinct points $x, y \in X$, there exist $U \in e^*R(X, x)$ and $V \in e^*R(X, y)$ such that $U \cap V = \emptyset$;

(7) For every pair of distinct points $x, y \in X$, there exist $U \in e^*\theta O(X, x)$ and $V \in e^*\theta O(X, y)$ such that $e^*-\text{cl}_\theta(U) \cap e^*-\text{cl}_\theta(V) = \emptyset$.

**Proof.** (3) $\Rightarrow$ (2) : Obvious.

(2) $\Rightarrow$ (1) : Obvious.

(1) $\Rightarrow$ (3) : Let $x, y \in X$ and $x \neq y$.

\[
(x, y \in X)(x \neq y) \implies (\exists W \in e^*\theta O(X, x))(y \notin W) \tag{1}
\]

\[
\text{Lemma } 2.4 \quad (\exists U \in e^*R(X, x))(U = e^*-\text{cl}_\theta(U) \subseteq W) \implies V := \setminus U = \setminus e^*-\text{cl}_\theta(U) \\
(\exists U \in e^*\theta O(X, x))(V \in e^*\theta O(X, y))(U \cap V = \emptyset). \tag{3} \]

(3) $\Rightarrow$ (4) : The proof is obvious since $e^*\theta O(X) \subseteq e^*O(X)$.

(4) $\Rightarrow$ (5) : Let $x, y \in X$ and $x \neq y$.

\[
(x, y \in X)(x \neq y) \implies (\exists U \in e^*O(X, x))(\exists V \in e^*O(X, y))(U \cap V = \emptyset) \tag{4}
\]

\[
\implies (\exists U \in e^*O(X, x))(\exists V \in e^*O(X, y))(U \subseteq \setminus V) \Rightarrow (\exists U \in e^*O(X, x))(\exists V \in e^*O(X, y))(e^*-\text{cl}(U) \subseteq \setminus V) \tag{5}
\]

\[
\implies (\exists U \in e^*O(X, x))(\exists V \in e^*O(X, y))(e^*-\text{int}(e^*-\text{cl}(U)) = e^*-\text{cl}(U) \subseteq e^*-\text{int}(\setminus V)) \\
\implies (\exists U \in e^*O(X, x))(\exists V \in e^*O(X, y))(e^*-\text{cl}(U) \subseteq e^*-\text{int}(\setminus V) = e^*-\text{cl}(V)) \\
\implies (\exists U \in e^*O(X, x))(\exists V \in e^*O(X, y))(e^*-\text{int}(U) \cap e^*-\text{cl}(V) = \emptyset). \tag{6}
\]

(5) $\Rightarrow$ (6) : Let $x, y \in X$ and $x \neq y$. 

ON ALMOST CONTRA $e^\ast \vartheta$-CONTINUOUS FUNCTIONS

$(x, y \in X)(x \neq y)$ \quad (5) \quad \Rightarrow \quad (\exists U_1 \in e^\ast O(X, x))(\exists V_1 \in e^\ast O(X, y))(e^\ast-cl(U_1) \cap e^\ast-cl(V_1) = \emptyset) \quad (6) \Rightarrow (7) : \text{Let } x, y \in X \text{ and } x \neq y.$

$$\Rightarrow (\exists V_1 \in e^\ast R(X, x))(\exists V_2 \in e^\ast R(X, y))(U_2 \cap V_2 = \emptyset).$$

$(6) \Rightarrow (7) : \text{Let } x, y \in X \text{ and } x \neq y.$

$(x, y \in X)(x \neq y)$ \quad (6) \quad \Rightarrow (\exists U \in e^\ast R(X, x))(\exists V \in e^\ast R(X, y))(U \cap V = \emptyset)$

$$\Rightarrow (\exists U \in e^\ast \theta O(X, x))(\exists V \in e^\ast \theta O(X, y))(e^\ast-cl(U) \cap e^\ast-cl(V) = \emptyset).$$

$(7) \Rightarrow (3) : \text{Obvious.} \quad \Box$

Definition 4.2. A topological space $X$ is said to be:

a) weakly Hausdorff [27] (briefly weakly-$T_2$) if every point of $X$ is an intersection of regular closed sets of $X$,

b) $s$-Urysohn [2] if for each pair of distinct points $x$ and $y$ in $X$, there exist $U \in SO(X, x)$ and $V \in SO(X, y)$ such that $cl(U) \cap cl(V) = \emptyset$.

Theorem 4.2. For a function $f : X \to Y$, the following properties hold:

(1) If $f$ is an almost contra $e^\ast \theta$-continuous injection of a topological space $X$ into a $s$-Urysohn space $Y$, then $X$ is $e^\ast \theta$-$T_2$,

(2) If $f$ is an almost contra $e^\ast \theta$-continuous injection of a topological space $X$ into a weakly Hausdorff space $Y$, then $X$ is $e^\ast \theta$-$T_1$.

Proof. (1) Let $x, y \in X$ and $x \neq y$.

$(x, y \in X)(x \neq y)$ \quad \Rightarrow \quad f(x) \neq f(y) \quad \Rightarrow \quad Y \text{ is } s\text{-Urysohn}$

$$\Rightarrow (\exists V_1 \in SO(Y, f(x)))(\exists V_2 \in SO(Y, f(y)))(cl(V_1) \cap cl(V_2) = \emptyset) \quad \text{Theorem 3.1(4)} \quad \Rightarrow \quad f \text{ is a.c. } e^\ast \theta\text{-c.}$$
\[ \Rightarrow (\exists U_1 \in e^\theta O(X, x))(\exists U_2 \in e^\theta O(X, y))(f[U_1] \cap f[U_2] \subseteq cl(V_1) \cap cl(V_2) = \emptyset) \]
\[ \Rightarrow (\exists U_1 \in e^\theta O(X, x))(\exists U_2 \in e^\theta O(X, y))(f[U_1] \cap f[U_2] = \emptyset) \]
\[ \Rightarrow (\exists U_1 \in e^\theta O(X, x))(\exists U_2 \in e^\theta O(X, y))(U_1 \cap U_2 = \emptyset). \]

(2) Let \( x, y \in X \) and \( x \neq y \).

\[
(x, y \in X)(x \neq y) \quad f \text{ is injective} \quad \Rightarrow f(x) \neq f(y) \quad Y \text{ is weakly-}T_2 \quad \Rightarrow
\]
\[
(\exists V_1 \in RC(Y, f(x)))(\exists V_2 \in RC(Y, f(y)))(f(x) \notin V_2)(f(y) \notin V_1) \quad \text{Theorem 3.1(3)} \quad \Rightarrow
\]
\[
(\exists U_1 \in e^\theta O(X, x))(\exists U_2 \in e^\theta O(X, y))(f[U_1] \subseteq V_1)(f[U_2] \subseteq V_2)(f(x) \notin V_2)(f(y) \notin V_1) \quad \Rightarrow
\]
\[
(\exists U_1 \in e^\theta O(X, x))(\exists U_2 \in e^\theta O(X, y))(x \notin U_2)(y \notin U_1). \quad \square
\]

Remark 2. [15] The intersection of two \( e^\theta \)-open sets is not necessarily \( e^\theta \)-open as shown in the following example.

**Example 4.1.** [15] Let \( X = \{a, b, c, d\} \) and \( \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \). Although the subsets \( \{b, c, d\} \) and \( \{a, c, d\} \) are \( e^\theta \)-open in \( X \), the set \( \{c, d\} \) which is the intersection of these sets is not \( e^\theta \)-open in \( X \).

**Definition 4.3.** A topological space \( X \) is called an \( e^\theta c \)-space if the intersection of any two \( e^\theta \)-open sets is an \( e^\theta \)-open set.

**Theorem 4.3.** If \( f, g : X \to Y \) are almost contra \( e^\theta \)-continuous functions, \( X \) is an \( e^\theta c \)-space and \( Y \) is \( s \)-Urysohn, then \( E = \{x \in X | f(x) = g(x)\} \) is \( e^\theta \)-closed in \( X \).

**Proof.** Let \( x \notin E \).

\[
x \notin E \Rightarrow f(x) \neq g(x) \quad Y \text{ is } s\text{-Urysohn} \quad \Rightarrow
\]
\[
(\exists V_1 \in SO(Y, f(x)))(\exists V_2 \in SO(Y, g(x)))(cl(V_1) \cap cl(V_2) = \emptyset) \quad f \text{ and } g \text{ are a.c.e}^\theta \text{c.} \quad \Rightarrow
\]
\[ \Rightarrow (\exists U_1 \in e^*\theta O(X, x))(\exists U_2 \in e^*\theta O(X, x))(f[U_1] \cap g[U_2] \subseteq \text{cl}(V_1) \cap \text{cl}(V_2) = \emptyset) \]

\[ X \text{ is e}^*\text{c-space} \\]

\[ \Rightarrow (\exists U := U_1 \cap U_2 \in e^*\theta O(X, x))(f[U] \cap g[U] \subseteq f[U_1] \cap g[U_2] = \emptyset) \]

\[ \Rightarrow (\exists U \in e^*\theta O(X, x))(U \cap E = \emptyset) \]

\[ \Rightarrow x \notin e^*\text{c}_\theta(E). \]

We say that the product space \( X = X_1 \times \ldots \times X_n \) has Property \( P_{e^*\theta} \) if \( A_i \) is an \( e^*\theta \)-open set in a topological space \( X_i \) for \( i = 1, 2, \ldots n \), then \( A_1 \times \ldots \times A_n \) is also \( e^*\theta \)-open in the product space \( X = X_1 \times \ldots \times X_n \).

**Theorem 4.4.** Let \( f : X_1 \to Y \) and \( g : X_2 \to Y \) be two functions, where

(i) \( X = X_1 \times X_2 \) has the Property \( P_{e^*\theta} \),

(ii) \( Y \) is a Urysohn space,

(iii) \( f \) and \( g \) are almost contra \( e^*\theta \)-continuous,

then \( A = \{ (x_1, x_2) | f(x_1) = g(x_2) \} \) is \( e^*\theta \)-closed in the product space \( X = X_1 \times X_2 \).

**Proof.** Let \( (x_1, x_2) \notin A \).

\[ (x_1, x_2) \notin A \Rightarrow f(x_1) \neq g(x_2) \]

\[ \Rightarrow (\exists V_1 \in O(Y, f(x_1)))(\exists V_2 \in O(Y, g(x_2)))(\text{cl}(V_1) \cap \text{cl}(V_2) = \emptyset)(\text{cl}(V_1), \text{cl}(V_2) \in RC(Y)) \]

\[ f \text{ and } g \text{ are a.e.e}^*\text{c}. \]

\[ \Rightarrow (f^{-1}[\text{cl}(V_1)] \in e^*\theta O(X_1, x_1))(g^{-1}[\text{cl}(V_2)] \in e^*\theta O(X_2, x_2)) \]

\[ X = X_1 \times X_2 \text{ has the Property } P_{e^*\theta} \]

\[ \Rightarrow ((x_1, x_2) \in f^{-1}[\text{cl}(V_1)] \times g^{-1}[\text{cl}(V_2)] \in e^*\theta O(X))(f^{-1}[\text{cl}(V_1)] \times g^{-1}[\text{cl}(V_2)] \subseteq \setminus A) \]

\[ \Rightarrow \setminus A \in e^*\theta O(X_1 \times X_2) \]

\[ A \in e^*\theta C(X_1 \times X_2). \]
Theorem 4.5. Let \( f : X \to Y \) be a function and \( g : X \to X \times Y \) the graph function, given by \( g(x) = (x, f(x)) \) for every \( x \in X \). If \( g \) is almost contra \( e^*\theta \)-continuous, then \( f \) is almost contra \( e^*\theta \)-continuous.

Proof. Let \( V \in RO(Y) \).

\[
V \in RO(Y) \Rightarrow X \times V \in RO(X \times Y) \Rightarrow g^{-1}[V] = g^{-1}[X \times V] \in e^*\theta C(X).
\]

We recall that for a function \( f : X \to Y \), the subset \( \{(x, f(x)) | x \in X\} \) of \( X \times Y \) is called the graph of \( f \) and is denoted by \( G(f) \).

Definition 4.4. A function \( f : X \to Y \) has an \( e^*\theta \)-closed graph if for each \( (x, y) \notin G(f) \), there exist \( U \in e^*\theta O(X, x) \) and \( V \in O(Y, y) \) such that \( (U \times V) \cap G(f) = \emptyset \).

Lemma 4.1. The graph \( G(f) \) of a function \( f : X \to Y \) is \( e^*\theta \)-closed if and only if for each \( (x, y) \notin G(f) \), there exist \( U \in e^*\theta O(X, x) \) and \( V \in O(Y, y) \) such that \( f[U] \cap V = \emptyset \).

Proof. Straightforward.

Theorem 4.6. Let \( X \) and \( Y \) be two topological spaces. If \( f : X \to Y \) is a function with an \( e^*\theta \)-closed graph, then \( \{f(x)\} = \cap \{cl(f[U]) | U \in e^*\theta O(X, x)\} \) for each \( x \) in \( X \).

Proof. Let \( G(f) \) be \( e^*\theta \)-closed. Suppose that there exists a point of \( x \) in \( X \) such that

\[
\{f(x)\} \neq \cap \{cl(f[U]) | U \in e^*\theta O(X, x)\}.
\]

\[
\{f(x)\} \neq \cap \{cl(f[U]) | U \in e^*\theta O(X, x)\} \Rightarrow (\exists y \in \cap \{cl(f[U]) | U \in e^*\theta O(X, x)\})(y \neq f(x))
\]

\[
\Rightarrow (\forall U \in e^*\theta O(X, x))(y \in cl(f[U]))((x, y) \notin G(f)) \Rightarrow G(f) \text{ is } e^*\theta \text{-closed}
\]

\[
\Rightarrow (\exists V \in O(Y, y))(y \in cl(f[U]))(\emptyset = f[U] \cap V = cl(f[U]) \cap V \neq \emptyset)
\]

This is a contradiction.
Theorem 4.7. If $f : X \rightarrow Y$ is almost contra $e^\theta$-continuous and $Y$ is Hausdorff, then $G(f)$ is $e^\theta$-closed.

Proof. Let $(x, y) \notin G(f)$.

$(x, y) \notin G(f) \Rightarrow y \neq f(x)$ \quad (Y \text{ is Hausdorff})

$\Rightarrow (f(x) \notin Y \setminus \text{cl}(V))(U \subseteq Y \setminus \text{cl}(V) \in RO(Y)) \Rightarrow f(x) \notin \text{ker}(U)$

$\Rightarrow x \notin f^{-1}[\text{ker}(U)] \quad \text{f is a.e.-c.e.} \quad \Rightarrow x \notin e^\theta-\text{cl}(f^{-1}[U])$

$V := \setminus e^\theta-\text{cl}(f^{-1}[U])$

$\Rightarrow (V \in e^\theta O(X, x))(U \in O(Y, y))(V \times U \subseteq \setminus G(f))$

$\Rightarrow (V \in e^\theta O(X, x))(U \in O(Y, y))(V \times U) \cap G(f) = \emptyset)$. \hfill \square

Theorem 4.8. If $f : X \rightarrow Y$ have an $e^\theta$-closed graph and injective, then $X$ is $e^\theta-T_1$.

Proof. Let $x_1, x_2 \in X$ and $x_1 \neq x_2$.

$(x_1, x_2 \in X)(x_1 \neq x_2) \quad \text{f is injective}$

$\Rightarrow f(x_1) \neq f(x_2) \Rightarrow (x_1, f(x_2)) \in (X \times Y) \setminus G(f)$ \quad $G(f)$ is $e^\theta$-closed

$\Rightarrow (\exists U \in e^\theta O(X, x_1))(\exists V \in O(Y, f(x_2)))(f[U] \cap V = \emptyset)$

$\Rightarrow (\exists U \in e^\theta O(X, x_1))(\exists V \in O(Y, f(x_2)))(U \cap f^{-1}[V] = \emptyset)$

$\Rightarrow (\exists U \in e^\theta O(X, x_1))(x_2 \notin U)$

Then $X$ is $e^\theta-T_0$. On the other hand, the notions of $e^\theta-T_0$ and $e^\theta-T_1$ are equivalent from Theorem 4.1. Thus $X$ is $e^\theta-T_1$. \hfill \square

Theorem 4.9. If $f : X \rightarrow Y$ has an $e^\theta$-closed graph and $X$ is an $e^\theta c$-space, then $f^{-1}[K]$ is $e^\theta$-closed for every compact subset $K$ of $Y$. 
Proof. Let $K$ be a compact subset of $Y$ and let $x \notin f^{-1}[K]$.

\[ x \notin f^{-1}[K] \Rightarrow f(x) \notin K \Rightarrow (\forall y \in K)(y \neq f(x)) \Rightarrow (x, y) \in (X \times Y) \setminus G(f) \]

\[ G(f) \text{ is } e^*\theta\text{-closed} \]

\[ \Rightarrow (\exists U_y \in e^*\theta O(X, x))(\exists V_y \in O(Y, y))(f[U_y] \cap V_y = \emptyset) \]

\[ \Rightarrow (\forall y \in K)\bigg( y \neq f(x) \bigg) \Rightarrow (x, y) \in (X \times Y) \setminus G(f) \]

\[ \Rightarrow (\exists A \subseteq O(Y))(K \subseteq \cup A) \]

\[ \Rightarrow (\forall y \in K)\bigg( y \neq f(x) \bigg) \Rightarrow (x, y) \in (X \times Y) \setminus G(f) \]

\[ A := \{ V_y | y \in K \} \]

\[ X \text{ is a.e.}^*e^*\theta \text{-space} \]

\[ x \notin e^*\theta \text{-int}(X \setminus f^{-1}[K]) \]

Lemma 2.3 $(7) \Rightarrow x \notin e^*\theta \text{-cl}(f^{-1}[K])$

\[ \Rightarrow x \notin f^{-1}[K]. \]

\[ \Box \]

Definition 4.5. A topological space $X$ is said to be:

a) strongly $e^*\theta C$-compact if every $e^*\theta$-closed cover of $X$ has a finite subcover (resp. $A \subseteq X$ is strongly $e^*\theta C$-compact if the subspace $A$ is strongly $e^*\theta C$-compact),

b) nearly compact [26] if every regular open cover of $X$ has a finite subcover.

Theorem 4.10. If $f : X \to Y$ is an almost contra $e^*\theta$-continuous surjection and $X$ is strongly $e^*\theta C$-compact, then $Y$ is nearly compact.

Proof. Let $B \subseteq RO(Y)$ and $Y = \cup B$.

\[ (B \subseteq RO(Y))(Y = \cup B) \Rightarrow (A := \{ f^{-1}[B] | B \in B \} \subseteq e^*\theta C(X))(X = \cup A) \]

\[ X \text{ is strongly } e^*\theta C\text{-compact} \]
\[ (\exists \mathcal{A}^* \subseteq \mathcal{A})(|\mathcal{A}^*| < \aleph_0)(X = \cup \mathcal{A}^*) \Rightarrow f \text{ is surjective} \]
\[ (\exists \mathcal{B}^* := \{ f[A] | A \in \mathcal{A}^* \} \subseteq \mathcal{B})(|\mathcal{B}^*| < \aleph_0)(Y = \cup \mathcal{B}^*). \]

We recall that a topological space \( X \) is said to be almost regular [25] if for each regular closed set \( F \) of \( X \) and each point \( x \in X \setminus F \), there exist disjoint open sets \( U \) and \( V \) such that \( F \subseteq V \) and \( x \in U \).

**Theorem 4.11.** If a function \( f : X \to Y \) is almost contra-e\(^*\theta\)-continuous and \( Y \) is almost regular, then \( f \) is almost e\(^*\theta\)-continuous.

**Proof.** Let \( x \in X \) and \( V \in O(Y, f(x)) \).
\[
(x \in X)(V \in O(Y, f(x))) \Rightarrow \text{Lemma 2.8} \Rightarrow \text{Y is almost regular} \Rightarrow (\exists W \in RO(Y, f(x)))(cl(W) \subseteq int(cl(V))) \Rightarrow \text{Theorem 3.1(3)} \Rightarrow f \text{ is a.c.e}\(^*\theta\).c.
\]
\[
(\exists U \in e^*\theta O(X, x))(f[U] \subseteq cl(W) \subseteq int(cl(V))). \qedhere
\]

**Definition 4.6.** The e\(^*\theta\)-frontier of a subset \( A \), denoted by \( Fr_{e^*\theta}(A) \), is defined as \( Fr_{e^*\theta}(A) = e^*-cl_\theta(A) \setminus e^*-int_\theta(A) \), equivalently \( Fr_{e^*\theta}(A) = e^*-cl_\theta(A) \cap e^*-cl_\theta(X \setminus A) \).

**Theorem 4.12.** The set of points \( x \in X \) on which \( f : X \to Y \) is not almost contra-e\(^*\theta\)-continuous is identical with the union of the e\(^*\theta\)-frontiers of the inverse images of regular closed sets of \( Y \) containing \( f(x) \).

**Proof.** Let \( A := \{ x | f \text{ is not a.c.e}\(^*\theta\).c. at } x \in X \} \).
\[
x \in A \Rightarrow f \text{ is not a.c.e}\(^*\theta\).c. at } x
\]
\[
(\forall V \in RC(Y, f(x)))(\exists U \in e^*\theta O(X, x))(f[U] \not\subseteq V)
\]
\[
(\forall V \in RC(Y, f(x)))(\forall U \in e^*\theta O(X, x))(U \cap (X \setminus f^{-1}[V]) \neq \emptyset)
\]
\[
(x \in f^{-1}[V])(x \in e^*-cl_\theta(X \setminus f^{-1}[V]) = X \setminus e^*-int_\theta(f^{-1}[V])
\]
\[
x \in Fr_{e^*\theta}(f^{-1}[V]).
\]
Then we have \( A \subseteq \bigcup \{ Fr_{e^*\theta}(f^{-1}[V])| V \in RC(Y, f(x)) \} \ldots (\ast) \)
\[ x \notin A \Rightarrow f \text{ is a.c.e}^*\theta \text{.c. at } x \]
\[ V \in RC(Y, f(x)) \]
\[ \Rightarrow x \in e^*\text{-int}_\theta(f^{-1}[V]) \]
\[ \Rightarrow x \notin Fr_{e^*\theta}(f^{-1}[V]) \]
\[ \Rightarrow x \notin \bigcup \{ Fr_{e^*\theta}(f^{-1}[V])| V \in RC(Y, f(x)) \} \]
Then we have \( \bigcup \{ Fr_{e^*\theta}(f^{-1}[V])| V \in RC(Y, f(x)) \} \subseteq A \ldots (\ast\ast) \)
\((\ast), (\ast\ast) \Rightarrow A = \bigcup \{ Fr_{e^*\theta}(f^{-1}[V])| V \in RC(Y, f(x)) \} . \)

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