SUBSETS IN TERMS OF $\Psi_H$

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Abstract. In this paper, we study the properties of $\Psi_HA$–sets and $\Psi_HC$–sets introduced by Kim and Min. Also, we characterize these sets in terms of strongly $\mu$–codense hereditary classes.

1. Introduction

A family $\mu$ of subsets of a nonempty set $X$ is called a generalized topology (GT) [1] if $\emptyset \in \mu$ and the arbitrary union of members of $\mu$ is again in $\mu$. The pair $(X, \mu)$ is called a generalized topological space (GTS) or simply a space. The elements of $\mu$ are called $\mu$–open sets and the complements of $\mu$–open sets are called $\mu$–closed sets. The largest $\mu$–open set contained in a subset $A$ of $X$ is denoted by $i_\mu(A)$ [1] and is called the $\mu$–interior of $A$. The smallest $\mu$–closed set containing $A$ is called the $\mu$–closure of $A$ and is denoted by $c_\mu(A)$ [1]. A GT $\mu$ is said to be a quasi-topology [4] on $X$ if $M, N \in \mu$ implies $M \cap N \in \mu$. A subset $A$ of a space is said to be $\mu$–preopen [2](resp. $\mu$–rare [3], $\mu$–$\alpha$–open [2], $\mu$–semiopen [2], $\mu$–$\beta$–open [2]) if $A \subset i_\mu c_\mu(A)$ (resp. $i_\mu c_\mu(A) = \emptyset$, $A \subset i_\mu c_\mu i_\mu(A)$, $A \subset c_\mu i_\mu(A)$, $A \subset c_\mu i_\mu c_\mu(A)$). The

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family of all \(\mu\)-preopen (resp. \(\mu\)-\(\alpha\)-open, \(\mu\)-semiopen) sets in \((X, \mu)\) is denoted by \(\pi(\mu)\) (resp. \(\alpha(\mu), \sigma(\mu)\)).

A hereditary class \(\mathcal{H}\) of \(X\) is a nonempty collection of subsets of \(X\) such that \(A \subset B, B \in \mathcal{H}\) implies \(A \in \mathcal{H}\) [3]. A hereditary class \(\mathcal{H}\) of \(X\) is an ideal [6] if \(A \cup B \in \mathcal{H}\) whenever \(A \in \mathcal{H}\) and \(B \in \mathcal{H}\). With respect to the generalized topology \(\mu\) of all \(\mu\)-open sets and a hereditary class \(\mathcal{H}\), for each subset \(A\) of \(X\), a subset \(A^*\) of \(X\) is defined by

\[
A^* = \{x \in X | M \cap A \notin \mathcal{H} \text{ for every } M \in \mu \text{ containing } x\} \quad [3].
\]

\(\beta = \{U - H | U \in \mu \text{ and } H \in \mathcal{H}\}\) is a basis for \(\mathcal{H}\). The following lemmas will be useful in the sequel and we use some of the results without mentioning it, when the context is clear.

**Lemma 1.1.** [3] Let \(X\) be a nonempty set and \(\mathcal{H}\) be a hereditary class on \(X\). If \(A\) and \(B\) are any two subsets of \(X\), then the following hold.

(a) If \(A \in \mathcal{H}\), then \(A^* = X - \mathcal{M}_\mu\) where \(\mathcal{M}_\mu = \bigcup\{M | M \in \mu\}\).

(b) If \(A \subset A^*\), then \(c_\mu(A) = A^* = c^*(A) = c^*(A^*)\).

(c) \(A^*\) is \(\mu\)-closed for every subset \(A\) of \(X\).
Lemma 1.2. [7, Theorem 2.4] If $(X, \mu)$ is a quasi-topological space and $\mathcal{H}$ is a hereditary class of subsets of $X$, then the following statements are equivalent.

(a) $\mathcal{H}$ is $\mu$-codense.

(b) $\mathcal{H}$ is strongly $\mu$-codense.

Lemma 1.3. [7, Theorem 2.5] If $X$ is a nonempty set, $\mathcal{H}$ is a hereditary class of subsets of $X$, then the following statements are equivalent.

(a) $\mathcal{H}$ is strongly $\mu$-codense.

(b) $M \subset M^\star$ for every $M \in \mu$.

(c) $c_\mu(M) = M^\star$ for every $M \in \mu$.

Lemma 1.4. [7, Theorem 2.6] If $(X, \mu)$ is a quasi-topological space and $\mathcal{H}$ is a hereditary class of subsets of $X$, then $M \cap A^\star \subset (M \cap A^\star)^\star$ for every $M \in \mu$ and $A \subset X$.

Lemma 1.5. [9, Theorem 2.7] Let $(X, \mu)$ be a quasi-topological space with a hereditary class $\mathcal{H}$ on $X$. Then the following are equivalent.

(a) $\pi(\gamma) \cap \mathcal{H} = \{\emptyset\}$.

(b) $A \subset A^\star$ for every subset $A \in \pi(\gamma)$.

(c) $i_\mu(A) = \emptyset$ for every $A \in \mathcal{H}$.

2. $\Psi_{\mathcal{H}}A$-SET

If $\mathcal{H}$ is a hereditary class on a space $(X, \mu)$, an operator $\Psi_\mathcal{H} : \wp(X) \to \wp(X)$ [8] is defined as follows: for every $A \in \wp(X)$, $\Psi_\mathcal{H}(A) = \{x \in X \mid$ there exists $M \in \mu$ containing $x$ such that $M - A \in \mathcal{H}\}$. $\Psi_\mathcal{H}$ is nothing but the monotonic operator $\gamma_\mu^* : \wp(X) \to \wp(X)$ defined by $\gamma_\mu^*(A) = X - (X - A)^\star$ for every subset $A$ of $X$ in [5]. A subset $A$ of $X$ is said to be $\Psi_{\mathcal{H}}A$-set if $A \subset i_\mu^*c_\mu \Psi_\mathcal{H}(A)$. Since $\Psi_\mathcal{H}(A)$ is $\mu$-open for every subset $A$ of $X$ and $M \subset \Psi_\mathcal{H}(M)$ for every $M \in \mu$ [5, Theorem 3.3], clearly every $\mu - \alpha$-open set is a $\Psi_{\mathcal{H}}A$-set.
2.1. Theorem 2.2(a) below gives a characterization of $\Psi$ which implies that $(U_{\mu}x/\alpha)$

Lemma 2.1. [8, Theorem 2.4] Let $(X, \mu)$ be a quasi-topological space and $\mathcal{H}$ be an ideal on $X$. If $A, B \subset X$, then $\Psi_{\mathcal{H}}(A \cap B) = \Psi_{\mathcal{H}}(A) \cap \Psi_{\mathcal{H}}(B)$.

Proof. Clearly, $\Psi_{\mathcal{H}}(A \cap B) \subseteq \Psi_{\mathcal{H}}(A) \cap \Psi_{\mathcal{H}}(B)$. Let $x \in \Psi_{\mathcal{H}}(A) \cap \Psi_{\mathcal{H}}(B)$. Since $x \notin (X - A)^*$, there exists $U_x \in \mu$ such that $U_x \cap (X - A) \in \mathcal{H}$ which implies that $U_x - A \in \mathcal{H}$. Since $x \notin (X - B)^*$, there exists $V_x \in \mu$ such that $V_x \cap (X - B) \in \mathcal{H}$ which in turn implies that $V_x - B \in \mathcal{H}$. Since $(V_x \cap U_x) - A \subset U_x - A, (V_x \cap U_x) - A \in \mathcal{H}$, by heredity. Similarly, $(U_x \cap V_x) - B \in \mathcal{H}$. Therefore, $((U_x \cap V_x) - A) \cup ((U_x \cap V_x) - B) \in \mathcal{H}$ which implies that $(U_x \cap V_x) \cap ((X - A) \cup (X - B)) \in \mathcal{H}$ and so $(U_x \cap V_x) \cap (X - (A \cap B)) \in \mathcal{H}$. Since $x \in U_x \cap V_x, x \notin (X - (A \cap B))^*$. Hence $x \in \Psi_{\mathcal{H}}(A \cap B)$. Hence $\Psi_{\mathcal{H}}(A \cap B) = \Psi_{\mathcal{H}}(A) \cap \Psi_{\mathcal{H}}(B)$.

The following Theorem 2.1 shows that the collection of all $\Psi_{\mathcal{H}}\mathcal{A}$-sets, denoted by $\mu_{\mathcal{A}}$, is a generalized topology, if $\mathcal{H}$ is a hereditary class. Example 2.1 below shows that the conditions quasi-topology on $\mu$ and ideal on $\mathcal{H}$ cannot be dropped in Theorem 2.1. Theorem 2.2(a) below gives a characterization of $\Psi_{\mathcal{H}}\mathcal{A}$-sets. Example 2.2 shows that a $\Psi_{\mathcal{H}}\mathcal{A}$-set need not be a $\mu$-semiopen set.

Theorem 2.1. Let $(X, \mu)$ be a space with a hereditary class $\mathcal{H}$. Then $\mu_{\mathcal{A}}$ is a generalized topology on $X$. Further, if $\mu$ is a quasi-topology and $\mathcal{H}$ is an ideal, then $\mu_{\mathcal{A}}$ is also a quasi-topology on $X$.

Proof. Clearly, $\emptyset \in \mu_{\mathcal{A}}$. Let $\{A_{\alpha} | \alpha \in \Delta\}$ be a family of $\Psi_{\mathcal{H}}\mathcal{A}$-sets in $(X, \mu)$. Then for each $\alpha \in \Delta$, $A_{\alpha} \subset i_{\mu}c_{\mu}\Psi_{\mathcal{H}}(A_{\alpha}) \subset i_{\mu}c_{\mu}\Psi_{\mathcal{H}}(\cup A_{\alpha})$ and so $\cup A_{\alpha} \subset i_{\mu}c_{\mu}\Psi_{\mathcal{H}}(\cup A_{\alpha})$. Hence $\cup A_{\alpha} \in \mu_{\mathcal{A}}$. Let $A$ and $B$ be $\Psi_{\mathcal{H}}\mathcal{A}$-sets in $X$. Then $A \cap B \subset i_{\mu}c_{\mu}\Psi_{\mathcal{H}}(A) \cap i_{\mu}c_{\mu}\Psi_{\mathcal{H}}(B) \subset i_{\mu}(c_{\mu}\Psi_{\mathcal{H}}(A) \cap i_{\mu}c_{\mu}\Psi_{\mathcal{H}}(B)) \subset i_{\mu}c_{\mu}(\Psi_{\mathcal{H}}(A) \cap \Psi_{\mathcal{H}}(B)) = i_{\mu}c_{\mu}\Psi_{\mathcal{H}}(A \cap B)$, by Lemma 2.1. Therefore, $A \cap B \in \mu_{\mathcal{A}}$.

Example 2.1. (a) Let $X = \{a, b, c, d\}, \mu = \{\emptyset, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ and $\mathcal{H} = \{\emptyset, \{c\}, \{d\}, \{c, d\}\}$. Then $i_{\mu}c_{\mu}\Psi_{\mathcal{H}}(\{a, c\}) = i_{\mu}c_{\mu}(\{a, c\}) = i_{\mu}(X) = \{a, b, c\} \supset \{a, c\}$.
and so \( \{a, c\} \in \mu_A \). Also, \( i_\mu c_\mu \Psi_H(\{b, c\}) = i_\mu c_\mu(\{b, c\}) = i_\mu(X) = \{a, b, c\} \supset \{b, c\} \) implies that \( \{b, c\} \in \mu_A \). But \( i_\mu c_\mu \Psi_H(\{c\}) = i_\mu c_\mu(\{\emptyset\}) = i_\mu(\{d\}) = \emptyset \not\supset \{c\} \). Hence \( \{c\} \notin \mu_A \).

(b) Let \( X = \{a, b, c, d\} \), \( \mu = \{\emptyset, \{c\}, \{a, b, c\}, \{c, d\}, X\} \) and \( \mathcal{H} = \{\emptyset, \{a\}, \{b\}\} \). Then \( i_\mu c_\mu \Psi_H(\{a, b, c\}) = i_\mu c_\mu(\{c, d\}) = i_\mu(X) = X \supset \{a, b, c\} \) which implies that \( \{a, b, c\} \in \mu_A \). Also, \( i_\mu c_\mu \Psi_H(\{a, b, d\}) = i_\mu c_\mu(\{a, b, c\}) = i_\mu(X) = X \supset \{a, b, d\} \) which implies that \( \{a, b, d\} \in \mu_A \). But \( i_\mu c_\mu \Psi_H(\{a, b\}) = i_\mu c_\mu(\{\emptyset\}) = i_\mu(\{d\}) = \emptyset \not\supset \{a, b\} \). Hence \( \{a, b\} \notin \mu_A \).

**Theorem 2.2.** Let \((X, \mu)\) be a space with a hereditary class \(\mathcal{H}\). Then the following hold.

(a) \( A \in \mu_A \) if and only if \( c_\mu i_\mu(X \setminus A)^* \subset X \setminus A \).

(b) \( \alpha(\mu) \subset \mu_A \).

(c) \( \mu^* \subset \mu_A \).

**Proof.** (a) \( A \in \mu_A \) if and only if \( A \subset i_\mu c_\mu \Psi_H(A) \) if and only if \( A \subset i_\mu c_\mu(X - (X - A)^*) \) if and only if \( A \subset X - c_\mu i_\mu(X - A)^* \) if and only if \( c_\mu i_\mu(X - A)^* \subset X - A \).

(b) \( A \in \alpha(\mu) \) implies that \( A \subset i_\mu c_\mu i_\mu(A) \) which implies that \( c_\mu i_\mu c_\mu(X - A) \subset X - A \).

Now \( c_\mu i_\mu(X - A)^* \subset c_\mu i_\mu c_\mu(X - A) \subset X - A \) and so \( A \in \mu_A \), by (a).

(c) Suppose \( A \in \mu^* \). Then by Theorem 3.18 of [5], \( A \subset \Psi_H(A) \). Now \( A \subset \Psi_H(A) = X - (X - A)^* \subset X - c_\mu i_\mu(X - A)^* = i_\mu c_\mu(X - (X - A)^*) = i_\mu c_\mu \Psi_H(A) \) which implies that \( A \subset i_\mu c_\mu \Psi_H(A) \) and so \( A \in \mu_A \). Thus, \( \mu^* \subset \mu_A \). \( \square \)

**Example 2.2.** Consider the GTS \((X, \mu)\) with a hereditary class \(\mathcal{H}\) where \( X = \{a, b, c\}, \mu = \{\emptyset, \{a, b\}, \{a, c\}, X\} \) and \( \mathcal{H} = \{\emptyset, \{a\}\} \). If \( A = \{b, c\} \), then \( i_\mu c_\mu \Psi_H(A) = i_\mu c_\mu(X) = X \) and so \( A \) is a \(\Psi_H \mathcal{A}-\)set. But \( c_\mu i_\mu(A) = c_\mu(\emptyset) = \emptyset \) implies that \( A \) is not \(\mu\)–semiopen.
The following Theorem 2.3 gives a characterization of strongly $\mu$–codense hereditary class in terms of $\Psi_H A$–sets. Example 2.3 below shows that the strongly $\mu$–codenseness on the hereditary class cannot be dropped in Theorem 2.4.

**Lemma 2.2.** [8, Theorem 2.14] Let $(X, \mu)$ be a space with a strongly $\mu$–codense ideal $\mathcal{H}$. Then $\Psi_H(A) \subset A^*$ for every subset $A$ of $X$. Moreover, if $A \in \mathcal{H}$, then $\Psi_H(A) = \emptyset$.

*Proof.* Suppose that $x \in \Psi_H(A)$ and $x \not\in A^*$. Since $x \in \Psi_H(A)$, there exists a $\mu$–open set $U$ containing $x$ such that $U - A \in \mathcal{H}$. Since $x \not\in A^*$, there is a $\mu$–open set $V$ containing $x$ such that $V \cap A \in \mathcal{H}$. Therefore, $(U \cap V) \cap A \in \mathcal{H}$ and $(U \cap V) - A \in \mathcal{H}$. By hypothesis, $\mathcal{H}$ is strongly $\mu$–codense and so $U \cap V = (U \cap V - A) \cup (U \cap V \cap A) \in \mathcal{H}$ implies that $U \cap V = \emptyset$, a contradiction to the fact that $x \in U \cap V$. Hence $x \in A^*$ so that $\Psi_H(A) \subset A^*$. Since $A \in \mathcal{H}$, by Lemma 1.1(a), $\Psi_H(A) \subset X - \mathcal{M}_\mu$ and so $\Psi_H(A) = \emptyset$. $\square$

**Theorem 2.3.** Let $(X, \mu)$ be a space with an ideal $\mathcal{H}$. Then the following are equivalent.

(a) $\mathcal{H}$ is strongly $\mu$–codense.

(b) $\mu A \subset \mathcal{H}O(\mu)$.

(c) $\mu^* \subset \mathcal{H}O(\mu)$.

*Proof.* (a)$\Rightarrow$(b). Suppose that $A \in \mu A$. Then by Lemma 2.2 and Lemma 1.1(c), $A \subset i_\mu c_\mu \Psi_H(A) \subset i_\mu c_\mu (A^*) = i_\mu (A^*)$ and so $A$ is $\mathcal{H}$–open.

(b)$\Rightarrow$(c). Follows from Theorem 2.2(c).

(c)$\Rightarrow$(a). Suppose $A$ is $\mu$–open. Then $A \in \mu^*$ and so $A \subset i_\mu (A^*)$, by hypothesis. Hence $A \subset A^*$ and so $\mathcal{H}$ is strongly $\mu$–codense. $\square$

**Lemma 2.3.** Let $(X, \mu)$ be a space with a hereditary class $\mathcal{H}$. Then the following hold.
Since \( \beta \)

**Theorem 2.4.**

Let

(a) If \( \mathcal{H} \) is strongly \( \mu \)-codense and \( A \subset X \) is \( \mu \)-closed, then \( i_\mu(A) = \Psi_\mathcal{H}(A) = i_\mu^*(A) \).

(b) For any subset \( A \) of \( X \), \( i_\mu^*(A) = A \cap \Psi_\mathcal{H}(A) \).

**Proof.** (a) Suppose that \( A \) is \( \mu \)-closed. Then by Lemma 1.3, \( c^*(X - A) = (X - A)^* = c_\mu(X - A) \) which implies that \( X - i_\mu^*(A) = (X - A)^* = X - i_\mu(A) \) which in turn implies that \( i_\mu^*(A) = \Psi_\mathcal{H}(A) = i_\mu(A) \).

(b) Let \( x \in A \cap \Psi_\mathcal{H}(A) \). Then \( x \in A \) and \( x \in \Psi_\mathcal{H}(A) \). Since \( x \in \Psi_\mathcal{H}(A) \), there exists \( M_x \in \mu \) containing \( x \) such that \( M_x - A \in \mathcal{H} \). Therefore, \( x \in M_x - (M_x - A) \subset A \). Since \( \beta \) is a basis for \( \mu^* \) and \( M_x - (M_x - A) \in \beta \), \( x \in i_\mu^*(A) \) where \( i_\mu^* \) is the interior operator in \( (X, \mu^*) \). Conversely, assume that \( x \in i_\mu^*(A) \). Then there exists a \( \mu^* \)-open set \( M_x \) containing \( x \) and \( H \in \mathcal{H} \) such that \( x \in M_x - H \subset A \). Now \( M_x - H \subset A \) implies that \( M_x - A \subset H \) which in turn implies that \( M_x - A \in \mathcal{H} \) and so \( x \in \mathcal{H}(A) \). Therefore, \( x \in A \cap \Psi_\mathcal{H}(A) \). Hence \( A \cap \Psi_\mathcal{H}(A) = i_\mu^*(A) \).

**Theorem 2.4.** Let \( (X, \mu) \) be a quasi-topological space with a \( \mu \)-codense ideal \( \mathcal{H} \). Then \( \mu_A = \alpha(\mu^*) \).

**Proof.** If \( A \in \mu_A \), then \( A \subset i_\mu c_\mu \Psi_\mathcal{H}(A) \). By Lemma 2.2, \( A \subset i_\mu c_\mu (\Psi_\mathcal{H}(A) \cap A^*) \) which implies that \( A \subset i_\mu c_\mu (\Psi_\mathcal{H}(A) \cap A^*) \subset i_\mu c_\mu^* (i_\mu^*(A)) \subset i_\mu^* c_\mu^* \Psi_\mathcal{H}(A) \), by Lemma 2.3(b). Thus, \( A \in \alpha(\mu^*) \) and so \( \mu_A \subset \alpha(\mu^*) \). Conversely, let \( A \in \alpha(\mu^*) \). Then \( A \subset i_\mu^* c_\mu^* (A) = i_\mu^* c_\mu^* (A \cap \Psi_\mathcal{H}(A)) \subset i_\mu^* c_\mu^* \Psi_\mathcal{H}(A) = i_\mu^* c_\mu \Psi_\mathcal{H}(A) \). Therefore, \( A \in \mu_A \). Hence \( \mu_A = \alpha(\mu^*) \).

**Example 2.3.** Consider the space \( (X, \mu) \) where \( X = \{a, b, c, d\} \), \( \mu = \{\emptyset, \{d\}, \{a, b, c\}, \{c, d\}, X\} \) and \( \mathcal{H} = \{\emptyset, \{c\}, \{d\}\} \). Clearly, \( \mathcal{H} \) is not strongly \( \mu \)-codense. Here \( \mu_A = \{\emptyset, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\} \) which is not a quasi-topology. If \( A = \{a, d\} \), then \( A \in \mu_A \). Since \( i_\mu^* c_\mu^* (\{d\}) = i_\mu^* (\{d\}) = \{d\} \notin \{a, d\} \), \( A \) is not \( \alpha \)-open in \( (X, \mu^*) \).
The following Theorem 2.5 gives characterizations of $\mu-$codense ideals in a quasi-topological space.

**Theorem 2.5.** Let $(X, \mu)$ be a quasi-topological space and $\mathcal{H}$ be an ideal. Then the following are equivalent.

(a) $\mathcal{H}$ is $\mu-$codense.

(b) $\mu_A \cap \mathcal{H} = \{\emptyset\}$.

(c) $A \subset A^*$ for $A \in \mu_A$.

**Proof.** (a)$\Rightarrow$(b). Suppose $A \in \mu_A \cap \mathcal{H}$. Then $A \in \mu_A$ and $A \in \mathcal{H}$. By Lemma 2.2, $A \in \mathcal{H}$ implies that $\Psi_{\mathcal{H}}(A) = \emptyset$. Since $A \in \mu_A$, $A \subset i_\mu c_\mu \Psi_{\mathcal{H}}(A) = i_\mu c_\mu(\emptyset) = i_\mu(X - \mathcal{M}_\mu) = \emptyset$ and so $A = \emptyset$.

(b)$\Rightarrow$(c). Let $A \in \mu_A$. Suppose $x \notin A^*$. Then there exists $M \in \mu$ containing $x$ such that $M \cap A \in \mathcal{H}$. Since $M \in \mu$, $M \in \mu_A$ and so $M \cap A \in \mu_A$, by Theorem 2.1. Hence $M \cap A = \emptyset$, which implies that $x \notin A$. Therefore, $A \subset A^*$ for $A \in \mu_A$.

(c)$\Rightarrow$(a). Let $A \in \mu \cap \mathcal{H}$. Then $A \in \mu$ implies that $A \subset A^*$, by (c). Also, by Lemma 1.1(a), $A \in \mathcal{H}$ implies that $A^* = X - \mathcal{M}_\mu$. Therefore, $A \subset X - \mathcal{M}_\mu$ so that $A \cap \mathcal{M}_\mu = \emptyset$ which implies $A = \emptyset$. Hence $\mathcal{H}$ is $\mu-$codense. 

**Theorem 2.6.** Let $(X, \mu)$ be a space with a strongly $\mu-$codense hereditary class $\mathcal{H}$ and $A \subset X$. Then $\Psi_{\mathcal{H}}(A) \neq \emptyset$ if and only if $i_\mu^*(A) \neq \emptyset$.

**Proof.** Suppose $\Psi_{\mathcal{H}}(A) \neq \emptyset$. Then there exists $\emptyset \neq M \in \mu$ such that $M - A \in \mathcal{H}$. If $M - A = P$ for some $P \in \mathcal{H}$, then $M - P \subset A$. Since $M \in \mu$ and $P \in \mathcal{H}$, $M - P \in \beta$. Therefore, $M - P \in \mu^*$ and so $A$ has nonempty $\mu^*$-interior. Conversely, suppose that $A$ has nonempty $\mu^*$-interior. If $x \in A$, then there exists $M \in \mu$ containing $x$ and $P \in \mathcal{H}$ such that $M - P \subseteq A$. Since $M - A \subset P$, $M - A \in \mathcal{H}$ and so $\Psi_{\mathcal{H}}(A) \neq \emptyset$. 

□
3. $\Psi_\mathcal{H}C$-SET

A subset $A$ of a space $(X, \mu)$ is said to be $\Psi_\mathcal{H}C$-set if $A \subset c_\mu \Psi_\mathcal{H}(A)$. We denote the family of all $\Psi_\mathcal{H}C$-sets in $(X, \mu)$ by $\Psi_\mathcal{H}C(X)$. Clearly, every $\Psi_\mathcal{H}A$-set is a $\Psi_\mathcal{H}C$-set. Every $\mu$-semiopen set is a $\Psi_\mathcal{H}C$-set and so every $\mu$-open set is a $\Psi_\mathcal{H}C$-set. But the converse need not be true as shown by the following Example 3.1.

**Example 3.1.** Consider the space $(X, \mu)$ with the hereditary class $\mathcal{H}$ where $X = \{a, b, c, d\}$, $\mu = \{\phi, \{a\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ and $\mathcal{H} = \{\phi, \{a\}, \{b\}\}$. Since $\Psi_\mathcal{H}(\{b\}) = X$, $\{b\}$ is a $\Psi_\mathcal{H}C$-set. But it is neither $\mu$-open nor $\mu$-semiopen.

The following Theorem 3.1 shows that $\Psi_\mathcal{H}C$-sets are $\mu - \beta$-open sets, if $\mathcal{H}$ is a strongly $\mu$-codense ideal. Example 3.2 below shows that the converse of Theorem 3.1 need not be true and Example 3.3 below shows that the condition strongly $\mu$-codense on $\mathcal{H}$ cannot be dropped.

**Theorem 3.1.** Let $(X, \mu)$ be a space with a strongly $\mu$-codense ideal $\mathcal{H}$. Then every $\Psi_\mathcal{H}C$-set is a $\mu - \beta$-open set.

**Proof.** If $A$ is a $\Psi_\mathcal{H}C$-set, then by Lemma 2.2, $A \subset c_\mu \Psi_\mathcal{H}(A) \subset c_\mu i_\mu(A^*) \subset c_\mu i_\mu c_\mu(A)$. Therefore, $A$ is a $\mu - \beta$-open set. \qed

**Example 3.2.** Let $X = \{a, b, c, d\}$ $\mu = \{\phi, \{a\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ and $\mathcal{H} = \{\phi, \{a\}, \{c\}, \{a, c\}\}$. Then $\mathcal{H}$ is a strongly $\mu$-codense ideal. If $A = \{a, c\}$, then $c_\mu i_\mu c_\mu(A) = c_\mu i_\mu(X) = c_\mu(\{a, b, c\}) \supset A$. Thus, $A \subset c_\mu i_\mu c_\mu(A)$ and so $A$ is $\mu - \beta$-open. Again, $c_\mu \Psi_\mathcal{H}(A) = c_\mu(X - X) = c_\mu(\emptyset) = \{d\} \not\subset A$. Hence $A$ is not a $\Psi_\mathcal{H}C$-set.

**Example 3.3.** Let $X = \{a, b, c, d\}$ $\mu = \{\phi, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $\mathcal{H} = \{\phi, \{b\}, \{c\}\}$. Here $\mathcal{H}$ is not a strongly $\mu$-codense hereditary class. If $A = \{a\}$, then $\Psi_\mathcal{H}(A) = \{a, b\}$ which implies that $c_\mu \Psi_\mathcal{H}(A) = X \supset A$. Therefore, $A$ is a $\Psi_\mathcal{H}C$-set. But $c_\mu i_\mu c_\mu(A) = c_\mu i_\mu(\{a, d\}) = c_\mu(\phi) = \{d\} \not\subset A$ and so $A$ is not a $\mu - \beta$-open set.
In [4], it is established that the intersection of a \( \mu - \alpha \)–open set with a \( \mu \)–semiopen set is a \( \mu \)–semiopen set. The following Theorem 3.2 is analogous to this result.

**Theorem 3.2.** Let \((X, \mu)\) be a quasi-topological space with an ideal \( \mathcal{H} \). Then the intersection of a \( \mu - \alpha \)–open set with a \( \Psi_\mathcal{H}C \)–set is a \( \Psi_\mathcal{H}C \)–set.

**Proof.** Let \( A \) be a \( \mu - \alpha \)–open set and \( B \) be a \( \Psi_\mathcal{H}C \)–set. Then \( A \subset i_\mu c_\mu i_\mu(A) \) and \( B \subset c_\mu \Psi_\mathcal{H}(B) \). Now \( A \cap B \subset i_\mu c_\mu i_\mu(A) \cap c_\mu \Psi_\mathcal{H}(B) \subset c_\mu(i_\mu c_\mu i_\mu(A) \cap \Psi_\mathcal{H}(B)) \subset c_\mu i_\mu i_\mu i_\mu i_\mu(A) \cap \Psi_\mathcal{H}(B) \subset c_\mu i_\mu i_\mu \Psi_\mathcal{H}(A) \cap \Psi_\mathcal{H}(B) = c_\mu \Psi_\mathcal{H}(A \cap B) \), by Lemma 2.1. Therefore, \( A \cap B \) is a \( \Psi_\mathcal{H}C \)–set. \( \square \)

The following Theorem 3.3 gives a characterization of \( \Psi_\mathcal{H}C \)– sets and Theorem 3.4 below characterizes strongly \( \mu \)–codense ideal in terms of \( \Psi_\mathcal{H}C \)–sets.

**Theorem 3.3.** Let \((X, \mu)\) be a space with a hereditary class \( \mathcal{H} \). Then the following are equivalent.
(a) \( A \) is a \( \Psi_\mathcal{H}C \)– set.
(b) \( A \subset X - i_\mu(X - A)^* \).
(c) \( A \subset c_\mu i_\mu \Psi_\mathcal{H}(A) \).

**Proof.** (a)\(\Leftrightarrow\)(b). \( A \) is a \( \Psi_\mathcal{H}C \)–set if and only if \( A \subset c_\mu \Psi_\mathcal{H}(A) \) if and only if \( A \subset c_\mu(X - (X - A)^*) \) if and only if \( A \subset X - i_\mu(X - A)^* \).

(a)\(\Leftrightarrow\)(c). Follows from the fact that \( \Psi_\mathcal{H}(A) \) is \( \mu \)–open. \( \square \)

**Theorem 3.4.** Let \((X, \mu)\) be a space with an ideal \( \mathcal{H} \). Then the following are equivalent.
(a) \( \mathcal{H} \) is strongly \( \mu \)–codense.
(b) For every \( \Psi_\mathcal{H}C \)–set \( A \), \( A \subset A^* \).
Proof. (a) ⇒ (b). Suppose that $A$ is a $\Psi_\mathcal{H}C$–set. Then $A \subset c_\mu \Psi_\mathcal{H}(A) \subset c_\mu(A^*) = A^*$, by Lemma 2.2 and Lemma 1.1(c) so $A \subset A^*$.

(b) ⇒ (a). Let $A$ be a $\mu$–open subset of $X$. Then $A$ is a $\Psi_\mathcal{H}C$–set and so $A \subset A^*$, by (b). Hence $\mathcal{H}$ is strongly $\mu$–codense, by Lemma 1.3.

The following Theorem 3.5 gives a characterization of $\mu$–codense hereditary class in terms of $\mathcal{H}$–open sets of a quasi-topological space and Theorem 3.6 shows that in a quasi-topological space $(X, \mu)$ with a $\mu$–codense ideal $\mathcal{H}$, the family of all $\Psi_\mathcal{H}C$–sets is nothing but the family of all $\mu^*$–semiopen sets in $X$. Corollary 3.1 below follows from the fact that $\sigma(\mu) \subset \sigma(\mu^*)$.

**Theorem 3.5.** Let $(X, \mu)$ be a quasi-topological space. Then the following are equivalent.

(a) $\mathcal{H}$ is (strongly) $\mu$–codense.

(b) $\mathcal{H}O(\mu) = \pi(\mu^*)$.

Proof. (a)⇒(b). Suppose $\mathcal{H}$ is (strongly) $\mu$–codense. Now, $A \in \mathcal{H}O(\mu)$ implies that $A \subset i_\mu(A^*) \subset i_\mu c_\mu^*(A) \subset i_\mu c_\mu^* c_\mu^*(A)$ and so $A \in \pi(\mu^*)$. If $A \in \pi(\mu^*)$, then $A \subset i_\mu c_\mu^*(A) \subset i_\mu c_\mu^* c_\mu^*(A) = i_\mu c_\mu^*(A)$ which implies that $A \in \pi(\mu)$. Therefore, $A \subset A^*$, by Lemma 1.5. Hence $A \subset i_\mu c_\mu^*(A)$ implies that $A \subset i_\mu(A^*)$ and so $A \in \mathcal{H}O(\mu)$.

(b)⇒(a). Suppose $A \in \mu$. Then $A \in \pi(\mu^*)$ and so $A \in \mathcal{H}O(\mu)$ which implies that $A \subset i_\mu(A^*) \subset A^*$. Hence $\mathcal{H}$ is (strongly) $\mu$–codense.

**Theorem 3.6.** Let $(X, \mu)$ be a quasi-topological space with a $\mu$–codense ideal $\mathcal{H}$. Then $\sigma(\mu^*) = \Psi_\mathcal{H}C(X)$.

Proof. Let $A \in \sigma(\mu^*)$. Then $A \subset c_\mu i_\mu^*(A) \subset c_\mu i_\mu^*(A) = c_\mu(A \cap \Psi_\mathcal{H}(A))$, by Theorem 2.3(b) and so $A \subset c_\mu \Psi_\mathcal{H}(A)$. Hence $\sigma(\mu^*) \subset \Psi_\mathcal{H}C(X)$. Conversely, let $A \in \Psi_\mathcal{H}C(X)$. If $x \notin c_\mu i_\mu^*(A)$, then $U \cap i_\mu^*(A) = \emptyset$ for some $\mu^*$–open set $U$ containing $x$ which implies that $(A \cap \Psi_\mathcal{H}(A)) \cap U = \emptyset$. Since $U \in \mu^*$, there exists $G \in \mu$ and $H \in \mathcal{H}$ such that
\(x \in G - H \subset U.\) Now \((A \cap \Psi_{H}(A)) \cap U = \emptyset\) implies that \(A \cap \Psi_{H}(A) \cap (G - H) = \emptyset\) which implies that \(A \cap \Psi_{H}(A) \cap G \subset H\) which in turn implies that \((A \cap \Psi_{H}(A) \cap G)^* \subset H^* = X - M_{\mu}\) and so \(A^* \cap \Psi_{H}(A) \cap G \subset X - M_{\mu},\) by Lemma 1.4. Thus, \(A^* \cap \Psi_{H}(A) \cap G = \emptyset\) and so \(\Psi_{H}(A) \cap G = \emptyset,\) by Lemma 2.2. Therefore, \(x \not\in c_{\mu} \Psi_{H}(A)\) so that \(x \not\in A.\) Thus, \(A \in \sigma(\mu^*)\) which implies that \(\Psi_{H} C(X) \subset \sigma(\mu^*).\)

**Corollary 3.1.** Let \((X, \mu)\) be a quasi-topological space with a \(\mu-\)codense ideal \(H.\) Then \(\sigma(\mu) \subset \Psi_{H} C(X).\)

**Corollary 3.2.** Let \((X, \mu)\) be a quasi-topological space with a \(\mu-\)codense ideal \(H.\) Then \(\mu_{A} = \mathcal{H}O(\mu) \cap \Psi_{H} C(X).\)

**Proof.** We know that \(\alpha(\mu) = \sigma(\mu) \cap \pi(\mu)[8].\) Since \(H\) is strongly \(\mu-\)codense and hence \(\mu-\)codense, by Theorem 3.5, \(\pi(\mu^*) = \mathcal{H}O(\mu)\) and by Theorem 3.6, \(\sigma(\mu^*) = \Psi_{H} C(X).\) Therefore, the proof follows from Theorem 2.4.

**Theorem 3.7.** Let \((X, \mu)\) be a quasi-topological space with an ideal \(H\) and \(A, B \subset X.\) If \(A \in \mu_{A},\) then \(A \cap B \in \Psi_{H} C(X)\) for every \(B \in \Psi_{H} C(X).\)

**Proof.** Let \(A \in \mu_{A}\) and \(B \in \Psi_{H} C(X).\) Then \(A \subset i_{\mu} c_{\mu} \Psi_{H}(A)\) and \(B \subset c_{\mu} \Psi_{H}(B).\) Suppose \(x \in A \cap B\) and \(U\) be a \(\mu-\)open set containing \(x.\) Since \(x \in A\) and \(A \subset i_{\mu} c_{\mu} \Psi_{H}(A),\) \(i_{\mu} c_{\mu} \Psi_{H}(A)\) is a \(\mu-\)open set containing \(x.\) Since \(\mu\) is a quasi-topology, \(U \cap i_{\mu} c_{\mu} \Psi_{H}(A)\) is also a \(\mu-\)open set containing \(x.\) Since \(\mu\) is a quasi-topology, \(U \cap i_{\mu} c_{\mu} \Psi_{H}(A)\) is also a \(\mu-\)open set containing \(x.\) Since \(x \in c_{\mu} \Psi_{H}(B)\), \(U \cap i_{\mu} c_{\mu} \Psi_{H}(A) \cap \Psi_{H}(B) \neq \emptyset.\) Let \(V = (U \cap i_{\mu} c_{\mu} \Psi_{H}(A)) \cap \Psi_{H}(B).\) Then \(V\) is an \(\mu-\)open set containing \(x\) such that \(V \subset i_{\mu} c_{\mu} \Psi_{H}(A) \neq c_{\mu} \Psi_{H}(A).\) Therefore, \(V \cap \Psi_{H}(A) \neq \emptyset\) which implies that \(U \cap i_{\mu} c_{\mu} \Psi_{H}(A) \cap \Psi_{H}(B) \cap \Psi_{H}(A) \neq \emptyset\) which in turn implies that \(U \cap \Psi_{H}(A \cap B) \neq \emptyset,\) by Lemma 2.1. Hence \(x \in c_{\mu} \Psi_{H}(A \cap B)\) and so \(A \cap B \subset c_{\mu} \Psi_{H}(A \cap B).\) Thus, \(A \cap B \in \Psi_{H} C(X).\)
Theorem 3.8. Let \((X, \mu)\) be a strong generalized space with a strong \(\mu\)-codense hereditary class \(\mathcal{H}\) and \(A, B \subset X\). If \(A \cap B \in \Psi_{\mathcal{H}}(X)\) for all \(B \in \Psi_{\mathcal{H}}(X)\), then \(A \in \mu_A\).

Proof. Since \(\emptyset \in \mathcal{H}\), by Lemma 1.1(a), \(c_\mu \Psi_{\mathcal{H}}(X) = c_\mu(M_\mu) = X\) which implies that \(X \in \Psi_{\mathcal{H}}(X)\) and so \(A \in \Psi_{\mathcal{H}}(X)\), by hypothesis. Suppose that \(x \in A\) and \(x \not\in i_\mu c_\mu \Psi_{\mathcal{H}}(A)\). Then \(x \in X - i_\mu c_\mu \Psi_{\mathcal{H}}(A) = c_\mu(X - c_\mu \Psi_{\mathcal{H}}(A))\). Let \(B = X - c_\mu \Psi_{\mathcal{H}}(A)\), then \(x \in c_\mu(B)\) so that \(V_x \cap B \neq \emptyset\) for every \(\mu\)-open set \(V_x\) containing \(x\). Since \(B\) is \(\mu\)-open, \(B \subset \Psi_{\mathcal{H}}(B)\) implies that \(V_x \cap B \subset V_x \cap \Psi_{\mathcal{H}}(B)\) and so \(V_x \cap \Psi_{\mathcal{H}}(B) \neq \emptyset\) which implies that \(x \in c_\mu \Psi_{\mathcal{H}}(B) \subset c_\mu \Psi_{\mathcal{H}}(\{x\} \cup B)\) implies that \(\{x\} \in c_\mu \Psi_{\mathcal{H}}(\{x\} \cup B)\). Also, \(B \subset c_\mu \Psi_{\mathcal{H}}(B)\) implies that \(B \subset c_\mu \Psi_{\mathcal{H}}(\{x\} \cup B)\). Hence \(\{x\} \cup B \subset c_\mu \Psi_{\mathcal{H}}(\{x\} \cup B)\). Therefore, \(\{x\} \cup B \in \psi_{\mathcal{H}} C(X)\). Therefore, by hypothesis, \(A \cap (\{x\} \cup B) \in \Psi_{\mathcal{H}} C(X)\). If possible, suppose there exists \(y \in X\) such that \(x \neq y\) and \(y \in A \cap (\{x\} \cup B)\). Then \(y \in A\) and \(y \in B\). Now \(y \in A\) implies that \(y \in c_\mu \Psi_{\mathcal{H}}(A)\), a contradiction to \(y \in B\). Therefore, \(A \cap (\{x\} \cup B) = \{x\}\) so that \(\{x\} \in \Psi_{\mathcal{H}} C(X)\).

Hence \(\{x\} \in c_\mu \Psi_{\mathcal{H}} \{x\}\). If \(\Psi_{\mathcal{H}} \{x\} = \emptyset\), then \(\{x\} \subset c_\mu \Psi_{\mathcal{H}} \{x\} \subset c_\mu \emptyset = \emptyset\), since \(\mu\) is strong. Hence \(\Psi_{\mathcal{H}} \{x\} \neq \emptyset\). Therefore, \(\{x\}\) contains a nonempty \(\mu^*-\)interior. Hence \(\{x\} = i_\mu^* \{x\} \subset i_\mu^* c_\mu \Psi_{\mathcal{H}} \{x\} = i_\mu \mu c_\mu \Psi_{\mathcal{H}} \{x\} = i_\mu c_\mu \Psi_{\mathcal{H}}(A \cap (\{x\} \cup B)) \subseteq i_\mu c_\mu \Psi_{\mathcal{H}}(A)\).

Therefore, \(x \in i_\mu c_\mu \Psi_{\mathcal{H}}(A)\), a contradiction to our assumption. Hence \(A \subset i_\mu c_\mu \Psi_{\mathcal{H}}(A)\) and so \(A \in \mu_A\). □

References


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