QUASI $\alpha grw$-OPEN MAPS IN TOPOLOGICAL SPACES

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Abstract. We introduce the notions of $\alpha$-generalized regular weakly open sets, Quasi $\alpha$-generalized regular weakly open maps and Quasi $\alpha$-generalized regular weakly closed maps in topological spaces.

1. Introduction

In 2010, A. Vadivel and K. Vairamanickam [5] introduced Quasi rw-open and Quasi rw-closed functions in topological spaces. In 2013, Varun Joshi et al. [6] introduced gprw-closed and gprw-Quasi closed functions in topological spaces. Recently, as a generalization of closed sets, the notion of $\alpha grw$-closed sets, $\alpha grw$-continuous maps and $\alpha grw$-open maps were introduced and studied in [2, 3]. In this paper we introduce and characterize the concept of quasi $\alpha grw$-open maps.

2. Preliminaries

Throughout this paper $X$, $Y$ and $Z$ denote topological spaces $(X, \tau)$, $(Y, \sigma)$ and $(Z, \eta)$ respectively on which no separation axioms are assumed unless otherwise mentioned. For a subset $A$ of a space $(X, \tau)$, $\text{cl}(A)$, $\text{int}(A)$, $\alpha \text{cl}(A)$, $\alpha \text{int}(A)$ and

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\( \alpha_{grw}\)-cl(A) denote the closure, interior, \( \alpha \)-closure, \( \alpha \)-interior and \( \alpha_{grw} \)-closure of A respectively. The complement of a set A of \((X, \tau)\) is denoted by \( A^c \) or \((X - A)\).

**Definition 2.1.** [4] A subset \( A \) of a topological space \((X, \tau)\) is called *regular open* if \( A = \text{int} (\text{cl}(A)) \).

**Definition 2.2.** [1] A subset \( A \) of a topological space \((X, \tau)\) is called regular semi-open if there is a regular open set \( U \) such that \( U \subseteq A \subseteq \text{cl}(U) \).

**Definition 2.3.** [2] A subset \( A \) of a topological space \((X, \tau)\) is called \( \alpha_{grw} \)-closed if \( \alpha_{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is regular semi-open.

**Definition 2.4.** [3] A map \( f : (X, \tau) \to (Y, \sigma) \) is called

1. \( \alpha_{grw} \)-continuous if \( f^{-1}(V) \) is an \( \alpha_{grw} \)-closed set of \((X, \tau)\) for every closed set \( V \) of \((Y, \sigma)\),
2. \( \alpha_{grw} \)-open if \( f(U) \) is \( \alpha_{grw} \)-open in \((Y, \sigma)\) for every open set \( U \) of \((X, \tau)\).

**Definition 2.5.** [3] For a subset \( A \) of \((X, \tau)\), \( \alpha_{grw}\)-cl(A)\( = \cap \{ F : A \subseteq F, F \) is \( \alpha_{grw} \)-closed in \( X \}\).

3. On \( \alpha \)-generalized regular weakly open sets

**Definition 3.1.** A subset \( A \) in \((X, \tau)\) is called \( \alpha \)-generalized regular weakly open (briefly \( \alpha_{grw} \)-open) if \( A^c \) is \( \alpha_{grw} \)-closed.

The family of \( \alpha_{grw} \)-open is denoted by \( \alpha_{grw}O(X, \tau) \) or \( \alpha_{grw}O(X) \).

**Theorem 3.1.** A subset \( A \) of \((X, \tau)\) is \( \alpha_{grw} \)-open if and only if \( F \subseteq \alpha_{int}(A) \) whenever \( F \) is regular semi-closed and \( F \subseteq A \).
Proof. Suppose that $F \subseteq \text{aint}(A)$, whenever $F$ is regular semi-closed and $F \subseteq A$. Let $A^c \subseteq U$, where $U$ is regular semi-open. Then $U^c \subseteq A$, where $U^c$ is regular semi-closed. By hypothesis $U^c \subseteq \text{aint}(A)$, which implies $[\text{aint}(A)]^c \subseteq U$. i.e., $\text{acl}(A^c) \subseteq U$. Thus $A^c$ is $\text{agrw}$-closed. Hence $A$ is $\text{agrw}$-open.

Conversely, suppose that $A$ is $\text{agrw}$-open, $F \subseteq A$ and $F$ is regular semi-closed. Then $F^c$ is regular semi-open and $A^c \subseteq F^c$. Therefore $\text{acl}(A^c) \subseteq F^c$ and so $F \subseteq [\text{acl}(A^c)]^c = \text{aint}(A)$. Hence $F \subseteq \text{aint}(A)$.

**Lemma 3.1.** If a subset $A$ of $X$ is $\text{agrw}$-closed, then $\text{acl}(A) - A$ does not contain any non-empty regular semi-closed set.

**Proof.** Suppose that $A$ is $\text{agrw}$-closed in $X$. Let $U$ be a regular semi-closed set such that $U \subseteq \text{acl}(A) - A$. Then $A \subseteq U^c$. Since $A$ is $\text{agrw}$-closed, we have $\text{acl}(A) \subseteq U^c$. Consequently, $U \subseteq [\text{acl}(A)]^c$. Thus $U \subseteq (\text{acl}(A)) \cap (\text{acl}(A))^c$. Hence $U = \emptyset$. Therefore $\text{acl}(A) - A$ does not contain any non-empty regular semi-closed set.

**Theorem 3.2.** If a subset $A$ is $\text{agrw}$-open in $(X, \tau)$, then $U = X$ whenever $U$ is regular semi-open and $\text{aint}(A) \cup A^c \subseteq U$.

**Proof.** Let $A$ be $\text{agrw}$-open, $U$ be regular semi-open such that $\text{aint}(A) \cup A^c \subseteq U$. This gives $U^c \subseteq [\text{aint}(A) \cup A^c]^c = \text{acl}(A^c) - A^c$. Since $A^c$ is $\text{agrw}$-closed and $U^c$ is regular semi-closed by Lemma 3.1, it follows that $U^c = \emptyset$. i.e., $X = U$.

**Theorem 3.3.** If $A$ is $\text{agrw}$-open and $\text{aint}(A) \subseteq B \subseteq A$, then $B$ is $\text{agrw}$-open.

**Proof.** Suppose that $\text{aint}(A) \subseteq B \subseteq A$ and $A$ is $\text{agrw}$-open. Then $A^c \subseteq B^c \subseteq \text{acl}(A^c)$ and since $A^c$ is $\text{agrw}$-closed, we have Theorem 3.26 [2], $B^c$ is $\text{agrw}$-closed i.e., $B$ is $\text{agrw}$-open.

**Definition 3.2.** Let $(X, \tau)$ be a topological space and $E \subseteq X$. $\text{agrw-int}(E)$ is the union of all $\text{agrw}$-open sets contained in $E$. 
i.e., $\alpha\text{grw-int}(E) = \bigcup \{ E : E \subseteq A \text{ and } E \text{ is } \alpha\text{grw-open} \}$.

Lemma 3.2. Let $A$ be a subset of a space $(X, \tau)$. Then $X - \alpha\text{grw-int}(A) = \alpha\text{grw-cl}(X - A)$.

Proof. Let $x \in X - \alpha\text{grw-int}(A)$. Then $x \notin \alpha\text{grw-int}(A)$. That is, every $\alpha\text{grw}$-open set $B$ containing $x$ is such that $B$ is not contained in $A$. This implies every $\alpha\text{grw}$-open set $B$ containing $x$ is such that $B \cap (X - A) \neq \emptyset$. By Theorem 4.15 [3], $x \in \alpha\text{grw-cl}(X - A)$. Hence $(X - \alpha\text{grw-int}(A)) \subseteq \alpha\text{grw-cl}(X - A)$.

Conversely let $x \in \alpha\text{grw-cl}(X - A)$ then by Theorem 4.15 [3], every $\alpha\text{grw}$-open set $B$ containing $x$ is such that $B \cap (X - A) \neq \emptyset$. That is every $\alpha\text{grw}$-open set $B$ containing $x$ is such that $B$ is not contained in $A$. This implies, $x \notin \alpha\text{grw-int}(A)$. Thus $x \in X - \alpha\text{grw-int}(A)$. Hence $\alpha\text{grw-cl}(X - A) \subseteq X - \alpha\text{grw-int}(A)$. Hence the proof.

Theorem 3.4. Let $(X, \tau)$ be a topological space. Then the following hold:

1. If $A \subseteq X$ is $\alpha\text{grw}$-closed, then $\alpha\text{cl}(A) - A$ is $\alpha\text{grw}$-open.
2. If $A$ is $\alpha\text{grw}$-open and $B$ is $\alpha\text{grw}$-open then $A \cap B$ is $\alpha\text{grw}$-open.
3. For any $E \subseteq X$, $\text{int}(E) \subseteq \alpha\text{grw-int}(E) \subseteq E$.

Proof.

1. Let $A$ be $\alpha\text{grw}$-closed. Let $F$ be regular semi-closed such that $F \subseteq \alpha\text{cl}(A) - A$. Then by Lemma 3.1, $F = \phi$. This implies $F \subseteq \alpha\text{int}(\alpha\text{cl}(A) - A)$. By Theorem 3.1, $\alpha\text{cl}(A) - A$ is $\alpha\text{grw}$-open.

2. Let $A^c$ and $B^c$ be $\alpha\text{grw}$-closed then $A^c \cup B^c$ is $\alpha\text{grw}$-closed by Theorem 3.19 [2]. This implies $A \cap B$ is $\alpha\text{grw}$-open.

3. Since every open set is $\alpha\text{grw}$-open, the proof follows immediately.

Definition 3.3. Let $(X, \tau)$ be a topological space. Let $\tau_{\alpha\text{grw}} = \{ U \subseteq X : \alpha\text{grw-cl}(X - U) = X - U \}$.
**Theorem 3.5.** Let \((X, \tau)\) be a topological space. Then the following hold:

1. Every \(\alpha grw\)-closed set is \(\alpha\)-closed if and only if \(\tau_{\alpha grw} = \alpha O(X, \tau)\).
2. Every \(\alpha grw\)-closed set is closed if and only if \(\tau_{\alpha grw} = \tau\).

**Proof.**

1. **Necessity.** Let \(A \in \tau_{\alpha grw}\). Then \(\alpha grw\)-cl\((X - A) = X - A\). By hypothesis, \(\alpha cl(X - A) \subseteq \alpha grw\)-cl\((X - A) = X - A\). This implies \(X - A\) is \(\alpha\)-closed and hence \(A \in \alpha O(X, \tau)\). Let \(A \in \alpha O(X, \tau)\), \(X - A\) is \(\alpha\)-closed which implies \(\alpha cl(X - A) = X - A\). Since every \(\alpha\)-closed set is \(\alpha grw\)-closed, \(\alpha grw\)-cl\((X - A) \subseteq \alpha cl(X - A) = X - A\). Hence \(A \in \tau_{\alpha grw}\).

2. **Sufficiency.** Suppose \(\tau_{\alpha grw} = \alpha O(X, \tau)\). Let \(A\) be \(\alpha grw\)-closed set. Then \(\alpha grw\)-cl\((A) = A\). This implies \(X - A \in \tau_{\alpha grw} = \alpha O(X, \tau)\). So \(A\) is \(\alpha\)-closed.

2. **Necessity.** Let \(A \in \tau_{\alpha grw}\). Then \(\alpha grw\)-cl\((X - A) = X - A\). By hypothesis, \(cl(X - A) \subseteq \alpha grw\)-cl\((X - A) = X - A\). This implies \(X - A\) is closed and hence \(A \in \tau\). Let \(A \in \tau\), \(X - A\) is closed which implies \(cl(X - A) = (X - A)\). Since every closed set is \(\alpha grw\)-closed, \(\alpha grw\)-cl\((X - A) \subseteq cl(X - A) = (X - A)\). Hence \(A \in \tau_{\alpha grw}\).

2. **Sufficiency.** Suppose \(\tau_{\alpha grw} = \tau\). Let \(A\) be \(\alpha grw\)-closed set. Then \(\alpha grw\)-cl\((A) = A\). This implies \(X - A \in \tau_{\alpha grw} = \tau\). So \(A\) is closed.

**Theorem 3.6.** If \(\alpha grw O(X, \tau)\) is closed under arbitrary union, then \(\tau_{\alpha grw}\) is a topology.

**Proof.**

1. Clearly, \(\emptyset, X \in \tau_{\alpha grw}\).
2. Let \(\{A_i : i \in \Lambda\} \in \tau_{\alpha grw}\). Then \(\alpha grw\)-cl\((X - (\bigcup A_i)) = \alpha grw\)-cl\((\bigcap (X - A_i)) \subseteq \bigcap \alpha grw\)-cl\((X - A_i) = \bigcap (X - A_i) = X - (\bigcup A_i)\). Therefore \(\bigcup A_i \in \tau_{\alpha grw}\).
3. Let \(A, B \in \tau_{\alpha grw}\). Now, \(\alpha grw\)-cl\((X - (A \cap B)) = \alpha grw\)-cl\((X - A) \cup \alpha grw\)-cl\((X - B) = (X - A) \cup (X - B) = X - (A \cap B)\). Then \(A \cap B \in \tau_{\alpha grw}\).

Hence \(\tau_{\alpha grw}\) is a topology.
4. Quasi $\alpha grw$-open maps

**Definition 4.1.** A map $f : (X, \tau) \to (Y, \sigma)$ is said to be quasi $\alpha grw$-open if the image of every $\alpha grw$-open set in $X$ is open in $Y$.

**Definition 4.2.** A map $f : (X, \tau) \to (Y, \sigma)$ is said to be quasi $\alpha grw$-closed if the image of every $\alpha grw$-closed set in $X$ is closed in $Y$.

It is evident that the concepts of quasi $\alpha grw$-openness (resp. $\alpha grw$-closedness) and $\alpha grw$-continuity coincide if the map is bijection.

**Theorem 4.1.** Every quasi $\alpha grw$-open map is open.

*Proof.* Let $U$ be an open set in $X$. Since every open set is $\alpha grw$-open, $U$ is $\alpha grw$-open in $X$. Then $f(U)$ is open in $Y$, since $f$ is quasi $\alpha grw$-open map. Hence $f$ is open.

**Corollary 4.1.** Every quasi $\alpha grw$-open map is $\alpha grw$-open.

*Proof.* It follows from Theorem 4.1 and Theorem 4.8(i) [3].

**Remark 1.** The converses of Theorem 4.1 and its corollary need not be true as seen from the following example.

**Example 4.1.** Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}\}$, $X$, $Y = \{p, q, r\}$ and $\sigma = \{\emptyset, \{p\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = f(b) = p$, $f(c) = q$ and $f(d) = r$. Then $f$ is open and $\alpha grw$-open but it is not quasi $\alpha grw$-open.

**Theorem 4.2.** A map $f : (X, \tau) \to (Y, \sigma)$ is said to be quasi $\alpha grw$-open if for every subset $U$ of $X$, $f(\alpha grw\text{-int}(U)) \subseteq \text{int}(f(U))$.

*Proof.* Assume that $U$ is $\alpha grw$-open set in $X$. Then $f(U) = f(\alpha grw\text{-int}(U)) \subseteq \text{int}(f(U))$ but $\text{int}(f(U)) \subseteq f(U)$. Consequently, $f(U) = \text{int}(f(U))$. Hence $f(U)$ is open in $Y$. Therefore $f$ is quasi $\alpha grw$-open.
Theorem 4.3. Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) be two maps and \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) be quasi \( \alpha grw \)-open. If \( g \) is continuous injective, then \( f \) is quasi \( \alpha grw \)-open.

Proof. Let \( U \) be a \( \alpha grw \)-open set in \( X \). Then, we have \((g \circ f)(U)\) is open in \( Z \), since \((g \circ f)\) is quasi \( \alpha grw \)-open. Again \( g \) is an injective continuous map, \( g^{-1}[(g \circ f)(U)] = f(U) \) is open in \( Y \). This shows that \( f \) is quasi \( \alpha grw \)-open.

Definition 4.3. A map \( f : (X, \tau) \rightarrow (Y, \sigma) \) is called \( \alpha grw^* \)-closed (resp. \( \alpha grw^* \)-open) map if the image of each \( \alpha grw \)-closed (resp. \( \alpha grw \)-open) subset in \( X \) is \( \alpha grw \)-closed (resp. \( \alpha grw \)-open) in \( Y \).

Theorem 4.4. For a topological space \((X, \tau)\), the following hold:

1. Every quasi \( \alpha grw \)-closed map is \( \alpha grw^* \)-closed.
2. Every \( \alpha grw^* \)-closed map is \( \alpha grw \)-closed.

Proof. Obvious.

Remark 2. The converses of the above theorem need not be true as seen from the following examples.

Example 4.2. Let \( X = \{a, b, c, d\} \), \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\} \), \( Y = \{p, q, r\} \) and \( \sigma = \{\emptyset, \{r\}, \{q, r\}, Y\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be defined by \( f(a) = f(b) = p \), \( f(c) = q \) and \( f(d) = r \). Then \( f \) is \( \alpha grw^* \)-closed but not quasi \( \alpha grw \)-closed.

Example 4.3. Let \( X = \{a, b, c, d\} \), \( \tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\} \), \( Y = \{p, q, r, s\} \) and \( \sigma = \{\emptyset, \{p\}, \{q\}, \{p, q\}, \{p, q, r\}, Y\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be defined by \( f(a) = s \), \( f(b) = q \), \( f(c) = p \) and \( f(d) = r \). Then \( f \) is \( \alpha grw \)-closed but not \( \alpha grw^* \)-closed.

Theorem 4.5. Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) be two maps on topological spaces.
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(1) If \( f \) is \( \alpha \text{grw} \)-closed and \( g \) is quasi \( \alpha \text{grw} \)-closed then \( g \circ f : (X, \tau) \to (Z, \eta) \) is closed.

(2) If \( f \) is quasi \( \alpha \text{grw} \)-closed and \( g \) is \( \alpha \text{grw} \)-closed then \( g \circ f : (X, \tau) \to (Z, \eta) \) is \( \alpha \text{grw}^* \)-closed.

(3) If \( f \) is \( \alpha \text{grw}^* \)-closed and \( g \) is quasi \( \alpha \text{grw} \)-closed then \( g \circ f : (X, \tau) \to (Z, \eta) \) is quasi \( \alpha \text{grw} \)-closed.

Proof. 1. Let \( F \) be a closed set in \( X \). Since \( f \) is \( \alpha \text{grw} \)-closed, \( f(F) \) is \( \alpha \text{grw} \)-closed set in \( Y \) and also \( g \) is a quasi \( \alpha \text{grw} \)-closed map therefore \( g(f(F)) \) is closed in \( Z \). Hence \( g \circ f \) is a closed map.

2. Let \( F \) be a \( \alpha \text{grw} \)-closed set in \( X \). Since \( f \) is quasi \( \alpha \text{grw} \)-closed, \( f(F) \) is closed set in \( Y \) and also \( g \) is an \( \alpha \text{grw} \)-closed map therefore \( g(f(F)) \) is \( \alpha \text{grw} \)-closed in \( Z \). Hence \( g \circ f \) is an \( \alpha \text{grw}^* \)-closed map.

3. Let \( F \) be a \( \alpha \text{grw} \)-closed set in \( X \). Since \( f \) is \( \alpha \text{grw}^* \)-closed, \( f(F) \) is \( \alpha \text{grw} \)-closed set in \( Y \) and also \( g \) is a quasi \( \alpha \text{grw} \)-closed map therefore \( g(f(F)) \) is closed in \( Z \). Hence \( g \circ f \) is a quasi \( \alpha \text{grw} \)-closed map.

Theorem 4.6. Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) be two maps such that \( g \circ f : (X, \tau) \to (Z, \eta) \) be a quasi \( \alpha \text{grw} \)-closed map. If \( g \) is \( \alpha \text{grw} \)-continuous and injective map then \( f \) is \( \alpha \text{grw}^* \)-closed.

Proof. Suppose that \( F \) is any \( \alpha \text{grw} \)-closed set in \( X \). Since \( g \circ f \) is quasi \( \alpha \text{grw} \)-closed therefore \( (g \circ f)(F) \) is closed in \( Z \) and also \( g \) is \( \alpha \text{grw} \)-continuous and injective map therefore \( g^{-1}[(g \circ f)(F)] = f(F) \), which is \( \alpha \text{grw} \)-closed in \( Y \). Hence \( f \) is \( \alpha \text{grw}^* \)-closed.

References


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