A Total Mean Cordial labeling of a graph $G = (V, E)$ is a function $f : V(G) \to \{0, 1, 2\}$ such that for each edge $xy$ assign the label $\left\lceil \frac{f(x) + f(y)}{2} \right\rceil$ where $x, y \in V(G)$ and $|ev_f(i) - ev_f(j)| \leq 1$, $i, j \in \{0, 1, 2\}$ where $ev_f(x)$ denotes the total number of vertices and edges labeled with $x$ ($x = 0, 1, 2$). If there exists a total mean cordial labeling on a graph $G$, we will call $G$ is Total Mean Cordial. In this paper, we investigate the Total Mean Cordial labeling behavior of fan, umbrella, dumbbell, and butterfly graphs.

1. Introduction

For standard terminology and notations in graph theory we refer the reader to Harary [4]. New terms and notations shall, however, be specifically defined whenever necessary. By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The number of vertices in $G$ is denoted by $|V(G)|$ and that of edges we denote $|E(G)|$. Graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Labeled graphs serve as useful models for a broad range of applications such as coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network addressing, database management, secret sharing schemes and models for constraint programming over finite domains,
for more details see [4]. Cahit [1] introduced the concept of cordial labeling and behaviors of cordial labeling have been studied by several authors, some of them are Diab, Riskin, Seoud, Abdel Maqsoud, Yousef [2, 9, 10, 11]. Ponraj, Ramasamy and Sathish Narayanan [5] introduced the concept of Total Mean Cordial labeling of graphs and studied about the Total Mean Cordial labeling behavior of path, cycle, wheel, lotus inside a circle, bistar, flower graph, \( K_{2,n} \), olive tree, the square of a path \( P_n^2 \), the corona of \( S(P_n) \) with \( K_1 \) that is \( S(P_n \odot K_1) \), \( S(K_{1,n}) \) and some more standard graphs in [7]. In [6, 8], Ponraj and Sathish Narayanan investigate the Total Mean Cordial behavior of one point union of 2 cycles \( C_n \) with a common vertex that is \( C_n^{(2)} \), ladder \( L_n \), book \( B_m \) and they proved that \( K_c^n + 2K_2 \) is Total Mean Cordial if and only if \( n = 1, 2, 4, 6, 8 \). In this paper we investigate the Total Mean Cordiality of fan, umbrella, dumbbell, and butterfly graphs. Let \( x \) be any real number. Then the symbol \( \lceil x \rceil \) stands for the smallest integer greater than or equal to \( x \).

2. Preliminary results

**Definition 2.1.** A Total Mean Cordial labeling of a graph \( G = (V,E) \) is a function \( f : V(G) \to \{0,1,2\} \) such that for each edge \( xy \) assign the label \( \lceil \frac{f(x)+f(y)}{2} \rceil \) where \( x, y \in V(G) \) and \( |ev_f(i)−ev_f(j)| \leq 1, i,j \in \{0,1,2\} \) where \( ev_f(x) \) denotes the total number of vertices and edges labeled with \( x \) \((x=0,1,2)\). If there exists a total mean cordial labeling on a graph \( G \), we will call \( G \) is Total Mean Cordial.

**Definition 2.2.** The join of two graphs \( G_1 \) and \( G_2 \) is denoted by \( G_1 + G_2 \) and whose vertex set is \( V(G_1 + G_2) = V(G_1) \cup V(G_2) \) and edge set \( E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\} \).

**Definition 2.3.** The graph \( C_n^{(t)} \) denotes the one point union of \( t \) copies of the cycle \( C_n : u_1 u_2 \ldots u_n u_1 \).
Definition 2.4. The graph $F_n = P_n + K_1$ is called a fan where $P_n : u_1 u_2 \ldots u_n$ be a path and $V(K_1) = u$.

Definition 2.5. The Umbrella $U_{n,m}$, $m > 1$ is obtained from a fan $F_n$ by pasting the end vertex of the path $P_m : v_1 v_2 \ldots v_m$ to the vertex of $K_1$ of the fan $F_n$.

Definition 2.6. Two even cycles of the same order, say $C_n$, sharing a common vertex with $m$ pendent edges attached at the common vertex is called a butterfly graph $B_{gm,n}$.

Definition 2.7. The graph obtained by joining two disjoint cycles $u_1 u_2 \ldots u_n u_1$ and $v_1 v_2 \ldots v_n v_1$ with an edge $u_1 v_1$ is called dumbbell graph $Db_n$.

Theorem 2.1. [5] Any path $P_m$ is Total Mean Cordial.

The above Theorem 2.1 is used for the investigation of the umbrella graph. So we recall the structure of the Total Mean Cordial labeling of path.

Let $P_m : v_1 v_2 \ldots v_m$ be the path. When $m = 3t$. Define a map $g : V(P_m) \rightarrow \{0, 1, 2\}$ by

$$
g(v_i) = 0, \quad 1 \leq i \leq t$$
$$
g(v_{t+i}) = 1, \quad 1 \leq i \leq t$$
$$
g(v_{2t+i}) = 2, \quad 1 \leq i \leq t.$$

When $m = 3t + 1$. The function $g : V(P_m) \rightarrow \{0, 1, 2\}$ is given below:

$$
g(v_i) = 0, \quad 1 \leq i \leq t + 1$$
$$
g(v_{t+1+i}) = 1, \quad 1 \leq i \leq t$$
$$
g(v_{2t+1+i}) = 2, \quad 1 \leq i \leq t.$$
When \( m = 3t + 2 \). The map \( g : V(P_m) \to \{0, 1, 2\} \) is given below:

\[
\begin{align*}
g(v_i) &= 0, \quad 1 \leq i \leq t + 1 \\
g(v_{t+1+i}) &= 1, \quad 1 \leq i \leq t \\
g(v_{2t+1+i}) &= 2, \quad 1 \leq i \leq t \\
g(v_{3t+2}) &= 1.
\end{align*}
\]

**Theorem 2.2.** [5] The cycle \( C_n \) is Total Mean Cordial if and only if \( n \neq 3 \).

The investigation of dumbbell graph is based on Theorem 2.2. So we recall the proof technique of Theorem 2.2. Let \( C_n : u_1 u_2 \ldots u_n u_1 \) be the cycle. If \( n = 3 \), then we have \( ev_f(0) = ev_f(1) = ev_f(2) = 2 \). But this is an impossible one. Assume \( n > 3 \).

When \( n \equiv 0 \mod 3 \), let \( n = 3t, \ t \in \mathbb{Z}^+ \). It is easy to see that \( C_6 \) is Total Mean Cordial. Take \( t \geq 3 \). Define \( f : V(C_n) \to \{0, 1, 2\} \) by

\[
\begin{align*}
f(u_i) &= 0, \quad 1 \leq i \leq t \\
f(u_{t+i}) &= 2, \quad 1 \leq i \leq t \\
f(u_{2t+i}) &= 1, \quad 1 \leq i \leq t - 2.
\end{align*}
\]

\( f(u_{3t-1}) = 0 \) and \( f(u_{3t}) = 1 \). When \( n \equiv 1, 2 \mod 3 \), The labeling \( f \) defined in case 2 of theorem 2.1 satisfy Total Mean Cordial condition of \( C_n \).

**Theorem 2.3.** [8] The graph \( C_n^{(2)} \) is Total Mean Cordial.

The investigation of butterfly graph is depends on the Total Mean Cordial labeling of \( C_n^{(2)} \). So we once again recall the Total Mean Cordiality of \( C_n^{(2)} \). Let the two copies of the cycles be \( u_1 u_2 \ldots u_n u_1 \) and \( v_1 v_2 \ldots v_n v_1 \). Join \( u_1 \) and \( v_1 \). It is easy to verify that \( C_n^{(2)} \) (\( 3 \leq n \leq 8 \)) is Total Mean Cordial. When \( n > 8 \). Assign the labels to the vertices of the two copies of the cycle \( C_n \) as in Theorem 2.2. If \( n \equiv 1 \mod 3 \) then relabel the vertices \( u_2, u_{t+2} \) and \( u_{t+4} \) by \( 2, 0, 0 \) respectively. If \( n \equiv 2 \mod 3 \) then relabel the vertex \( u_{t+2} \) by 0.
3. Main results

Now we look into the Fan graph.

**Theorem 3.1.** The fan $F_n$ is Total Mean Cordial if and only if $n \neq 2, n \neq 4$.

*Proof.* Clearly $|V(F_n)| + |E(F_n)| = 3n$.

**Case 1.** $n \equiv 0 \pmod{6}$.

Let $n = 6k$ where $k \in \mathbb{Z}^+$. Define a function $f : V(F_n) \to \{0, 1, 2\}$ by $f(u) = 0,$

\[
\begin{align*}
    f(u_i) &= 0, \quad 1 \leq i \leq 2k \\
    f(u_{2k+i}) &= 2, \quad 1 \leq i \leq 3k \\
    f(u_{5k+i}) &= 1, \quad 1 \leq i \leq k.
\end{align*}
\]

In this case $ev_f(0) = ev_f(1) = ev_f(2) = 6k$.

**Case 2.** $n \equiv 1 \pmod{6}$.

If we label $u_1$ and $u$ by 0 and 2, respectively, then $F_1$ is Total Mean Cordial. When $n = 7$, Figure 1 shows that $F_7$ is Total Mean Cordial.

![Figure 1](image)

Now, let $n = 6k + 1$ where $k \in \mathbb{Z}^+$ and $k \neq 0$. Define a map $f : V(F_n) \to \{0, 1, 2\}$ by $f(u) = f(u_{3k+2}) = 1,$

\[
\begin{align*}
    f(u_i) &= 0, \quad 1 \leq i \leq 3k + 1 \\
    f(u_{3k+2+i}) &= 2, \quad 1 \leq i \leq 2k \\
    f(u_{5k+2+i}) &= 1, \quad 1 \leq i \leq k - 1.
\end{align*}
\]

Here $ev_f(0) = ev_f(1) = ev_f(2) = 6k + 1$. 
Case 3. \( n \equiv 2 \pmod{6} \).

**Subcase 3.1.** \( n \equiv 2 \pmod{12} \). Obviously, \( F_2 \) is not Total Mean Cordial. Assume \( n > 2 \). Let \( n = 12k + 2 \) where \( k \in \mathbb{Z}^+ \). Define \( f : V(F_n) \to \{0, 1, 2\} \) by \( f(u) = 2, \ f(u_{12k+2}) = 0, \ f(u_{6k+2}) = 1, \)

\[
\begin{align*}
f(u_i) &= 0, \quad 1 \leq i \leq 6k + 1 \\
f(u_{6k+2+i}) &= 2, \quad 1 \leq i \leq 3k \\
f(u_{9k+2+i}) &= 1, \quad 1 \leq i \leq 3k - 1.
\end{align*}
\]

Here \( ev_f(0) = ev_f(1) = ev_f(2) = 12k + 2 \).

**Subcase 3.2.** \( n \equiv 8 \pmod{12} \). Let \( n = 12k - 4 \) where \( k \in \mathbb{Z}^+ \). Define \( f : V(F_n) \to \{0, 1, 2\} \) by \( f(u) = 2, \ f(u_{12k-4}) = 0, \)

\[
\begin{align*}
f(u_i) &= 0, \quad 1 \leq i \leq 6k - 2 \\
f(u_{6k-2+i}) &= 2, \quad 1 \leq i \leq 3k - 1 \\
f(u_{9k-3+i}) &= 1, \quad 1 \leq i \leq 3k - 2.
\end{align*}
\]

Here \( ev_f(0) = ev_f(1) = ev_f(2) = 12k - 4 \).

**Case 4.** \( n \equiv 3 \pmod{6} \).

When \( n = 3 \), Figure 2 shows that \( F_3 \) is Total Mean Cordial.

**Figure 2**

Let \( n = 6k + 3 \) where \( k \in \mathbb{Z}^+ \). Define a map \( f : V(F_n) \to \{0, 1, 2\} \) by \( f(u) = 0, \ f(u_{6k+3}) = 1, \)

\[
\begin{align*}
f(u_i) &= 0, \quad 1 \leq i \leq 2k + 1 \\
f(u_{2k+1+i}) &= 1, \quad 1 \leq i \leq k \\
f(u_{3k+1+i}) &= 2, \quad 1 \leq i \leq 3k + 1.
\end{align*}
\]

In this case \( ev_f(0) = ev_f(1) = ev_f(2) = 6k + 3 \).
Case 5. \( n \equiv 4 \pmod{6} \).

Subcase 5.1. \( n \equiv 4 \pmod{12} \). Then we have the following claim:

Claim. \( F_4 \) is not Total Mean Cordial.

Proof of the Claim: Suppose \( f(u) = 0 \). In this case at least two zeros should be labeled in the path vertices. This forces \( ev_f(0) \geq 5 \), a contradiction.

Suppose \( f(u) = 1 \). In this case, zero should be labeled to \( u_1, u_4 \) and \( u_2 \) (or \( u_3 \)). Then \( ev_f(2) = 2 \), a contradiction.

Suppose \( f(u) = 2 \). As in above zero should be labeled to \( u_1, u_4 \) and \( u_2 \) (or \( u_3 \)). In this case, \( ev_f(2) = 2 \) or 3, a contradiction. Proof of the claim is complete.

Let \( n = 12k + 4 \) where \( k \in \mathbb{Z}^+ \). Define \( f : V(F_n) \to \{0, 1, 2\} \) by \( f(u) = 2 \), \( f(u_{12k+4}) = 0 \),

\[
\begin{align*}
  f(u_i) &= 0, \quad 1 \leq i \leq 6k + 2 \\
  f(u_{6k+2+i}) &= 2, \quad 1 \leq i \leq 3k + 1 \\
  f(u_{9k+3+i}) &= 1, \quad 1 \leq i \leq 3k.
\end{align*}
\]

In this case \( ev_f(0) = ev_f(1) = ev_f(2) = 12k + 4 \).

Subcase 5.2. \( n \equiv 10 \pmod{12} \). Let \( n = 12k - 2 \) where \( k \in \mathbb{Z}^+ \). Define a function \( f : V(F_n) \to \{0, 1, 2\} \) by \( f(u) = 2 \), \( f(u_{6k}) = 1 \), \( f(u_{12k-2}) = 0 \),

\[
\begin{align*}
  f(u_i) &= 0, \quad 1 \leq i \leq 6k - 1 \\
  f(u_{6k+i}) &= 2, \quad 1 \leq i \leq 3k - 1 \\
  f(u_{9k-1+i}) &= 1, \quad 1 \leq i \leq 3k - 2.
\end{align*}
\]

Here \( ev_f(0) = ev_f(1) = ev_f(2) = 12k - 2 \).

Case 6. \( n \equiv 5 \pmod{6} \).
Subcase 6.1. \( n \equiv 5 \pmod{12} \). Let \( n = 12k - 7 \) where \( k \in \mathbb{Z}^+ \). Define \( f : V(F_n) \to \{0, 1, 2\} \) by \( f(u) = 2 \),

\[
\begin{align*}
f(u_i) &= 0, \quad 1 \leq i \leq 6k - 3 \\
f(u_{6k-3+i}) &= 2, \quad 1 \leq i \leq 3k - 2 \\
f(u_{9k-5+i}) &= 1, \quad 1 \leq i \leq 3k - 2.
\end{align*}
\]

In this case \( ev_f(0) = ev_f(1) = ev_f(2) = 12k - 7 \).

Subcase 6.2. \( n \equiv 11 \pmod{12} \). Let \( n = 12k - 1 \) where \( k \in \mathbb{Z}^+ \). Define a function \( f : V(F_n) \to \{0, 1, 2\} \) by \( f(u) = 2 \), \( f(u_{6k+1}) = 1 \),

\[
\begin{align*}
f(u_i) &= 0, \quad 1 \leq i \leq 6k \\
f(u_{6k+1+i}) &= 2, \quad 1 \leq i \leq 3k - 1 \\
f(u_{9k+i}) &= 1, \quad 1 \leq i \leq 3k - 1.
\end{align*}
\]

Here \( ev_f(0) = ev_f(1) = ev_f(2) = 12k - 1 \).

Hence \( F_n \) is Total Mean Cordial if and only if \( n \neq 2 \), \( n \neq 4 \).

\[\square\]

Next is the Umbrella graph.

**Theorem 3.2.** The Umbrella \( U_{n,m} \), \( m > 1 \) is Total Mean Cordial.

**Proof.** Take the vertex set and edge set of \( F_n \) as in Theorem 3.1. Further unify the vertices \( u \) and \( v_1 \). It is clear that \( |V(U_{n,m})| + |E(U_{n,m})| = 3n + 2m - 2 \). Let \( f, g \) respectively be the labellings defined for \( F_n, P_m \) in theorems 3.1, 2.1. Define a map \( h : V(U_{n,m}) \to \{0, 1, 2\} \) by \( h(u) = f(u) \),

\[
\begin{align*}
h(u_i) &= f(u_i), \quad 1 \leq i \leq n \\
h(v_i) &= g(v_i), \quad 2 \leq i \leq m.
\end{align*}
\]

**Case 1.** \( n \equiv 0 \pmod{6} \) and \( m \equiv 0 \pmod{3} \).

Let \( n = 6k \) and \( m = 3t \) where \( k, t \in \mathbb{Z}^+ \). When \( m = 3 \), relabel the vertices \( v_2, v_3 \) by 2, 0 respectively. Then \( ev_h(0) = ev_h(2) = 6k + 1, ev_h(1) = 6k + 2 \). For \( m = 6 \), relabel
the vertices \( v_4, v_5, v_6 \) by 2, 1, 0 respectively. In this case \( ev_h(0) = ev_h(2) = 6k + 3 \), 
\( ev_h(1) = 6k + 4 \). Assume \( m > 6 \). Now, relabel the vertex \( v_{t+2} \) by 0. Then 
\( ev_h(0) = ev_h(1) = 6k + 2t - 1, ev_h(2) = 6k + 2t \).

**Case 2.** \( n \equiv 0 \pmod{6} \) and \( m \equiv 1 \pmod{3} \).

Let \( n = 6k \) and \( m = 3t + 1 \) where \( k, t \in \mathbb{Z}^+ \). In this case 
\( ev_h(0) = ev_h(1) = ev_h(2) = 6k + 2t \).

**Case 3.** \( n \equiv 0 \pmod{6} \) and \( m \equiv 2 \pmod{3} \).

Let \( n = 6k \) and \( m = 3t + 2 \) where \( k, t \in \mathbb{Z}^+ \). In this case 
\( ev_h(0) = 6k + 2t, ev_h(1) = ev_h(2) = 6k + 2t + 1 \).

**Case 4.** \( n \equiv 1 \pmod{6} \) and \( m \equiv 0 \pmod{3} \).

Let \( n = 6k + 1 \) and \( m = 3t \) where \( k, t \in \mathbb{Z}^+ \).

**Subcase 4a.** \( n = 1 \). In this case \( U_{n,m} \cong P_{m+1} \). Then by theorem 2.1, \( P_{m+1} \) is Total Mean Cordial.

**Subcase 4b.** \( n = 7 \). Here relable \( h(v_{m-i+1}) = g(v_i), \ 1 \leq i \leq m \). Then 
\( ev_h(0) = ev_h(2) = 2t + 6, ev_h(1) = 2t + 7 \).

**Subcase 4c.** \( n \neq 1, n \neq 7 \). For \( m = 3 \), relabel the vertices \( v_2, v_3 \) by 2, 0 respectively. 
Then \( ev_h(0) = ev_h(1) = 6k + 2 \) and \( ev_h(2) = 6k + 3 \). When \( m = 6 \), relabel the vertices 
\( v_2, v_3, v_4, v_5, v_6 \) by 1, 2, 2, 0, 0 respectively. Then 
\( ev_h(0) = ev_h(1) = 6k + 4, ev_h(2) = 6k + 5 \). When \( m > 6 \), relabel the vertex \( v_{t+2} \) by 0. Then 
\( ev_h(0) = ev_h(1) = 6k + 2t, ev_h(2) = 6k + 2t + 1 \).

**Case 5.** \( n \equiv 1 \pmod{6} \) and \( m \equiv 1 \pmod{3} \).

Let \( n = 6k + 1 \) and \( m = 3t + 1 \) where \( k, t \in \mathbb{Z}^+ \).

**Subcase 5a.** \( n = 1 \). Similar to subcase 4a.

**Subcase 5b.** \( n = 7 \). For \( m = 4 \), relabel the vertices \( u_5, u_7, v_2, v_4 \) by 0, 2, 2, 1 respectively. Here 
\( ev_h(0) = ev_h(1) = ev_h(2) = 9 \). When \( m = 7 \), relabel \( u_5, u_6, v_2, \)
\( v_3, v_4, v_6 \) by 0, 0, 2, 2, 2, 1 respectively. In this case 
\( ev_h(0) = ev_h(1) = ev_h(2) = 11 \). When \( m > 7 \), relabel the vertex \( v_{t+3} \) by 0. Then 
\( ev_h(0) = ev_h(1) = ev_h(2) = 2t + 7 \).
Subcase 5c. $n \neq 1, n \neq 7$. For $m = 4$, relabel the vertices $u_{3k+2}$, $v_2$, $v_3$, $v_4$ by 0, 2, 1, 1, respectively. Then $ev_h(0) = ev_h(1) = ev_h(2) = 6k + 3$. For $m = 7$, relabel the vertices $v_2$, $v_3$, $v_4$, $v_5$, $v_6$, $v_7$ by 2, 2, 0, 1, 0, 0, respectively. Here, $ev_h(0) = ev_h(1) = ev_h(2) = 6k + 5$. When $m > 7$, relabel $v_{t+3}$ by 0. Then $ev_h(0) = ev_h(1) = ev_h(2) = 6k + 2t + 1$.

Case 6. $n \equiv 1 \pmod{6}$ and $m \equiv 2 \pmod{3}$.

Let $n = 6k + 1$ and $m = 3t + 2$ where $k, t \in \mathbb{Z}^+$.

Subcase 6a. $n = 1$. Similar to subcase 4a.

Subcase 6b. $n = 7$. For $m = 2$, relabel the vertex $v_2$ by 0. Then $ev_h(0) = ev_h(1) = 8, ev_h(2) = 7$. For $m = 5$, relabel $v_3$, $v_5$ by 2, 0 respectively. Here $ev_h(0) = 9, ev_h(1) = ev_h(2) = 10$. If $m > 5$, relabel $h(v_{m-i+1}) = g(v_i), 1 \leq i \leq m$. Then $ev_h(0) = ev_h(1) = 2t + 8, ev_h(2) = 2t + 7$.

Subcase 6c. $n \neq 1, n \neq 7$. Relabel the vertex $u_{3k+2}$ by 0. Here, $ev_h(0) = ev_h(1) = 6k + 2t + 2, ev_h(2) = 6k + 2t + 1$.

Case 7. $n \equiv 2 \pmod{6}$ and $m \equiv 0 \pmod{3}$.

Let $m = 3t$ where $t \in \mathbb{Z}^+$.

Subcase 7a. $n \equiv 2 \pmod{12}$. Let $n = 12k + 2$ where $k \in \mathbb{Z}^+$. Suppose $n = 2$. Then assign the labels 0, 2 to the vertices $u_1$, $u_2$ respectively. Then $ev_h(0) = ev_h(2) = 2t + 1, ev_h(1) = 2t + 2$. Assume $n > 2$. Here relabel $h(v_{m-i+1}) = g(v_i), 1 \leq i \leq m$. Then $ev_h(0) = ev_h(2) = 12k + 2t + 1, ev_h(1) = 12k + 2t + 2$.

Subcase 7b. $n \equiv 8 \pmod{12}$. Let $n = 12k - 4$ where $k \in \mathbb{Z}^+$. Here also relabel $h(v_{m-i+1}) = g(v_i), 1 \leq i \leq m$. Then $ev_h(0) = ev_h(2) = 12k + 2t - 5, ev_h(1) = 12k + 2t - 4$.

Case 8. $n \equiv 2 \pmod{6}$ and $m \equiv 1 \pmod{3}$.

Let $m = 3t + 1$ where $t \in \mathbb{Z}^+$.

Subcase 8a. $n \equiv 2 \pmod{12}$. Let $n = 12k + 2$ where $k \in \mathbb{Z}^+$. Suppose $n = 2$. Then relabel $h(v_{m-i+1}) = g(v_i), 1 \leq i \leq m$ and assign the labels 0, 2 to $u_1$, $u_2$.
respectively. Then $ev_h(0) = ev_h(1) = ev_h(2) = 2t + 2$. Suppose $n > 2$ and $m = 4$. Let $n = 12k + 2$. Define a function $f : V(U_{n,4}) \rightarrow \{0,1,2\}$ by $f(v_1) = f(v_4) = 0, f(v_2) = f(v_3) = 1,$

$$f(u_i) = 0, \quad 1 \leq i \leq 4k + 1$$
$$f(u_{4k+1+i}) = 2, \quad 1 \leq i \leq 6k + 2$$
$$f(u_{10k+3+i}) = 1, \quad 1 \leq i \leq 2k - 1.$$

In this case, $ev_f(0) = ev_f(1) = ev_f(2) = 12k + 4$. When $m = 7$, relabel the vertices $v_4, v_5$ by 2, 0 respectively. Here $ev_h(0) = ev_h(1) = ev_h(2) = 12k + 6$. For $m > 7$, relabel the vertex $v_{t+3}$ by 0. Then $ev_h(0) = ev_h(1) = ev_h(2) = 12k + 2t + 2$.

**Subcase 8b.** $n \equiv 8 \pmod{12}$. Let $n = 12k - 4$ where $k \in \mathbb{Z}^+$. For $m = 4$, let $n = 12k - 4$ and $k > 0$. Define a map $f : V(U_{n,4}) \rightarrow \{0,1,2\}$ by $f(v_1) = f(v_4) = 0, f(v_2) = 2, f(v_3) = 1,$

$$f(u_i) = 0, \quad 1 \leq i \leq 4k - 1$$
$$f(u_{4k-1+i}) = 2, \quad 1 \leq i \leq 6k - 2$$
$$f(u_{10k-3+i}) = 1, \quad 1 \leq i \leq 2k - 1.$$

In this case, $ev_f(0) = ev_f(1) = ev_f(2) = 12k - 2$. For $m \geq 7$, assign the labels to the vertices of $U_{n,m}$ as in subcase 8a. It is easy to check that $U_{n,m}$ is Total Mean Cordial.

**Case 9.** $n \equiv 2 \pmod{6}$ and $m \equiv 2 \pmod{3}$.

Let $m = 3t + 2$ where $t \in \mathbb{Z}^+$.

**Subcase 9a.** $n \equiv 2 \pmod{12}$. Let $n = 12k + 2$ where $k \in \mathbb{Z}^+$. Suppose $n = 2, m = 2$. Then assign the labels 0, 2 to $u_1, u_2$ respectively and 0, 2 to $v_1, v_2$ respectively. Then $ev_f(0) = ev_f(1) = 3, ev_f(2) = 2$. For $m > 2$, assign the labels 0, 2 to $u_1, u_2$ then $ev_h(0) = ev_h(1) = 2t + 3, ev_h(2) = 2t + 2$. Assume $n > 2$. When $m = 2$, assign the label 0 to $v_2$. In this case $ev_h(0) = ev_h(1) = 12k + 3, ev_h(2) = 12k + 2$. For $m > 2$, relabel $h(v_{m-i+1}) = g(v_i), \quad 2 \leq i \leq m$. Then $ev_h(0) = ev_h(1) = 12k + 2t + 3, ev_h(2) = 12k + 2t + 2$. 
Subcase 9b. $n \equiv 8 \pmod{12}$. Let $n = 12k - 4$ where $k \in \mathbb{Z}^+$. For $m = 2$, assign the label 0 to $v_2$. Then $ev_h(0) = ev_h(1) = 12k - 3$, $ev_h(2) = 12k - 4$. When $m > 2$, relabel $h(v_{m-i+1}) = g(v_i)$, $2 \leq i \leq m$. Then $ev_h(0) = ev_h(1) = 12k + 2t - 3$, $ev_h(2) = 12k + 2t - 4$.

Case 10. $n \equiv 3 \pmod{6}$ and $m \equiv 0 \pmod{3}$.

Let $n = 6k + 3$ and $m = 3t$ where $k, t \in \mathbb{Z}^+$. For $m = 3$, relabel $v_2$ by 0. Then $ev_h(0) = 6k + 5$, $ev_h(1) = ev_h(2) = 6k + 4$. For $m = 6$, relabel $v_3, v_6$ by 0, 1 respectively. In this case $ev_h(0) = 6k + 7$, $ev_h(1) = ev_h(2) = 6k + 6$. When $m > 6$, relabel $v_{i+2}$ by 0. In this case $ev_h(0) = ev_h(1) = 6k + 2t + 2$, $ev_h(2) = 6k + 2t + 3$.

Case 11. $n \equiv 3 \pmod{6}$ and $m \equiv 1 \pmod{3}$.

Let $n = 6k + 3$ and $m = 3t + 1$ where $k, t \in \mathbb{Z}^+$. In this case $ev_h(0) = ev_h(1) = ev_h(2) = 6k + 2t + 3$.

Case 12. $n \equiv 3 \pmod{6}$ and $m \equiv 2 \pmod{3}$.

Let $n = 6k + 3$ and $m = 3t + 2$ where $k, t \in \mathbb{Z}^+$. Here $ev_h(0) = 6k + 2t + 3$, $ev_h(1) = ev_h(2) = 6k + 2t + 4$.

Case 13. $n \equiv 4 \pmod{6}$ and $m \equiv 0 \pmod{3}$.

Let $m = 3t$ where $t \in \mathbb{Z}^+$.

Subcase 13a. $n \equiv 4 \pmod{12}$. Let $n = 12k + 4$ where $k \in \mathbb{Z}^+$. For $n = 4$, assign the labels 0, 2, 2, 0 to $u_1, u_2, u_3, u_4$ respectively. Then $ev_h(0) = ev_h(2) = 2t + 3$, $ev_h(1) = 2t + 4$. If $n > 4$ then relabel $h(v_{m-i+1}) = g(v_i)$, $2 \leq i \leq m$. Here $ev_h(0) = ev_h(2) = 12k + 2t + 3$, $ev_h(1) = 12k + 2t + 4$.

Subcase 13b. $n \equiv 10 \pmod{12}$. Let $n = 12k - 2$ where $k \in \mathbb{Z}^+$. Here also relabel $h(v_{m-i+1}) = g(v_i)$, $2 \leq i \leq m$. Then $ev_h(0) = ev_h(2) = 12k + 2t - 3$, $ev_h(1) = 12k + 2t - 2$.

Case 14. $n \equiv 4 \pmod{6}$ and $m \equiv 1 \pmod{3}$.

Let $m = 3t + 1$ where $t \in \mathbb{Z}^+$. 
Subcase 14a. $n \equiv 4 \pmod{12}$. Let $n = 12k + 4$ where $k \in \mathbb{Z}^+$. For $n = 4$, relabel $h(v_{m-i+1}) = g(v_i), \ 1 \leq i \leq m$. Also assign the labels 0, 0, 2, 1 to $u_1, u_2, u_3, u_4$ respectively. Then $ev_h(0) = ev_h(1) = ev_h(2) = 2t + 4$. For $n > 4$ and $m = 4$, define a function $\phi : V(U_{n,4}) \rightarrow \{0, 1, 2\}$ by $\phi(v_1) = 0, \phi(v_2) = \phi(v_3) = \phi(v_4) = 1$,

$$\phi(u_i) = 0, \ 1 \leq i \leq 4k + 2$$
$$\phi(u_{4k+2+i}) = 2, \ 1 \leq i \leq 6k + 3$$
$$\phi(u_{10k+5+i}) = 1, \ 1 \leq i \leq 2k - 1.$$

In this case, $ev_\phi(0) = ev_\phi(1) = ev_\phi(2) = 12k + 6$. When $n > 4$ and $m = 7$, relabel the vertices $v_2, v_4, v_6, v_7$ by 2, 2, 0, 0 respectively. Then $ev_h(0) = ev_h(1) = ev_h(2) = 12k + 8$. For $n > 7$ and $m > 7$, relabel the vertex $v_{t+3}$ by 0. In this case $ev_h(0) = ev_h(1) = ev_h(2) = 12k + 2t + 4$.

Subcase 14b. $n \equiv 10 \pmod{12}$. Let $n = 12k - 2$ where $k \in \mathbb{Z}^+$. For $m = 4$, define a map $\phi : V(U_{n,4}) \rightarrow \{0, 1, 2\}$ by $\phi(v_1) = 0, \phi(v_2) = 2, \phi(v_3) = \phi(v_4) = 1$,

$$\phi(u_i) = 0, \ 1 \leq i \leq 4k$$
$$\phi(u_{4k+i}) = 2, \ 1 \leq i \leq 6k - 1$$
$$\phi(u_{10k-1+i}) = 1, \ 1 \leq i \leq 2k - 1.$$

In this case, $ev_\phi(0) = ev_\phi(1) = ev_\phi(2) = 12k$. When $m = 7$, relabel the vertices $u_5, u_7$ by 2, 0 respectively. Here, $ev_h(0) = ev_h(1) = ev_h(2) = 12k + 2$. For $m > 7$, relabel the vertex $v_{t+3}$ by 0. In this case $ev_h(0) = ev_h(1) = ev_h(2) = 12k + 2t - 2$.

Case 15. $n \equiv 4 \pmod{6}$ and $m \equiv 2 \pmod{3}$.

Let $m = 3t + 2$ where $t \in \mathbb{Z}^+$.

Subcase 15a. $n \equiv 4 \pmod{12}$. Let $n = 12k + 4$ where $k \in \mathbb{Z}^+$. When $n = 4$, assign the labels 0, 2, 2, 0 to $u_1, u_2, u_3, u_4$, respectively. Then $ev_h(0) = ev_h(1) = 2t + 5, ev_h(2) = 2t + 4$. Assume $n > 4$. For $m = 2$, assign the label 0 to $v_2$. Then $ev_h(0) = ev_h(1) = 12k + 5, ev_h(2) = 12k + 4$. For $m > 2$, relabel $h(v_{m-i+1}) = g(v_i), \ 2 \leq i \leq m$. Then $ev_h(0) = ev_h(1) = 12k + 2t + 5, ev_h(2) = 12k + 2t + 4$. 


Subcase 15b. \( n \equiv 10 \pmod{12} \). Let \( n = 12k - 2 \) where \( k \in \mathbb{Z}^+ \). When \( m = 2 \), assign the label 0 to \( v_2 \). Here \( ev_h(0) = ev_h(1) = 12k - 1 \), \( ev_h(2) = 12k - 2 \). For \( m > 2 \), relabel \( h(v_{m-i+1}) = g(v_i), \ 2 \leq i \leq m \). Then \( ev_h(0) = ev_h(1) = 12k + 2t - 1 \), \( ev_h(2) = 12k + 2t - 2 \).

Case 16. \( n \equiv 5 \pmod{6} \) and \( m \equiv 0 \pmod{3} \).

Let \( m = 3t \) where \( t \in \mathbb{Z}^+ \).

Subcase 16a. \( n \equiv 5 \pmod{12} \). Let \( n = 12k - 7 \) where \( k \in \mathbb{Z}^+ \). Relabel \( h(v_{m-i+1}) = g(v_i), \ 2 \leq i \leq m \). In this case \( ev_h(0) = ev_h(2) = 12k + 2t - 8 \), \( ev_h(1) = 12k + 2t - 7 \).

Subcase 16b. \( n \equiv 11 \pmod{12} \). Let \( n = 12k - 1 \) where \( k \in \mathbb{Z}^+ \). Similar to subcase 16a. In this case, \( ev_h(0) = ev_h(2) = 12k + 2t - 2 \), \( ev_h(1) = 12k + 2t - 1 \).

Case 17. \( n \equiv 5 \pmod{6} \) and \( m \equiv 1 \pmod{3} \).

Let \( m = 3t + 1 \) where \( t \in \mathbb{Z}^+ \).

Subcase 17a. \( n \equiv 5 \pmod{12} \). Let \( n = 12k - 7 \) where \( k \in \mathbb{Z}^+ \). Suppose \( m = 4 \). Let \( n = 12k + 5 \) and \( k \geq 0 \). Define \( \phi : V(U_{n,4}) \to \{0, 1, 2\} \) by \( \phi(v_1) = \phi(v_3) = 0 \), \( \phi(v_2) = 2 \), \( \phi(v_4) = 1 \),

\[
\begin{align*}
\phi(u_i) &= 0, \quad 1 \leq i \leq 4k + 2 \\
\phi(u_{4k+2+i}) &= 2, \quad 1 \leq i \leq 6k + 3 \\
\phi(u_{10k+5+i}) &= 1, \quad 1 \leq i \leq 2k.
\end{align*}
\]

In this case, \( ev_{\phi}(0) = ev_{\phi}(1) = ev_{\phi}(2) = 12k + 7 \). For \( m = 7 \), relabel the vertices \( v_4, v_5 \) by 2, 0 respectively. Then \( ev_h(0) = ev_h(1) = ev_h(2) = 12k - 3 \). When \( m > 7 \), relabel the vertex \( v_{t+3} \) by 0. In this case \( ev_h(0) = ev_h(1) = ev_h(2) = 12k + 2t - 7 \).

Subcase 17b. \( n \equiv 11 \pmod{12} \). Let \( n = 12k - 1 \) where \( k \in \mathbb{Z}^+ \). When \( m = 4 \), let \( n = 12k - 1, k > 0 \). Define a map \( \phi : V(U_{n,4}) \to \{0, 1, 2\} \) by \( \phi(v_1) = \phi(v_3) = 0 \), \( \phi(v_2) = 2 \), \( \phi(v_4) = 1 \),
\[ \phi(u_i) = 0, \ 1 \leq i \leq 4k \]
\[ \phi(u_{4k+i}) = 2, \ 1 \leq i \leq 6k \]
\[ \phi(u_{10k+i}) = 1, \ 1 \leq i \leq 2k - 1. \]

In this case, \( ev_\phi(0) = ev_\phi(1) = ev_\phi(2) = 12k + 1. \) For \( m = 7, \) relabel the vertices \( v_4, v_5 \) by \( 2, 0 \) respectively. Here \( ev_h(0) = ev_h(1) = ev_h(2) = 12k + 3. \) For \( m > 7, \) relabel the vertex \( v_{t+3} \) by \( 0. \) In this case \( ev_h(0) = ev_h(1) = ev_h(2) = 12k + 2t - 1. \)

**Case 18.** \( n \equiv 5 \pmod{6} \) and \( m \equiv 2 \pmod{3}. \)

Let \( m = 3t + 2 \) where \( t \in \mathbb{Z}^+. \)

**Subcase 18a.** \( n \equiv 5 \pmod{12}. \) Let \( n = 12k - 7 \) where \( k \in \mathbb{Z}^+. \) For \( m = 2, \) relabel the vertex \( v_2 \) by \( 0. \) Then \( ev_h(0) = ev_h(1) = 12k - 6, ev_h(2) = 12k - 7. \) Assume \( m > 2. \) Then relabel \( h(v_{m-i+1}) = g(v_i), \ 2 \leq i \leq m. \) Here \( ev_h(0) = ev_h(1) = 12k + 2t - 6, ev_h(2) = 12k + 2t - 7. \)

**Subcase 18b.** \( n \equiv 11 \pmod{12}. \) Similar to subcase 18a.

Therefore, \( U_{n,m}, m > 1 \) is Total Mean Cordial. \( \square \)

**Example 3.1.** A Total Mean Cordial labeling of \( U_{9,7} \) is given in Figure 3.
We now investigate the butterfly graph.

**Theorem 3.3.** $B_{y_m,n}$ is Total Mean Cordial.

*Proof.* Assign the labels to the vertices of $C^{(2)}_n$ as in theorem 2.3.

**Case 1.** $n \equiv 0 \pmod{3}$ and $m \equiv 0 \pmod{3}$.

Let $m = 3k$ and $n = 3t$ where $t, k \in \mathbb{Z}^+$. Assign the label 0 to the first $k$ vertices and 2 to the next $2k$ vertices. Here, $ev_h(0) = 4t + 2k - 1$, $ev_h(1) = ev_h(2) = 4t + 2k$.

**Case 2.** $n \equiv 0 \pmod{3}$ and $m \equiv 1 \pmod{3}$.

Let $m = 3k + 1$ and and $n = 3t$ where $t, k \in \mathbb{Z}^+$. Assign the label 0 to the $k + 1$ vertices and 2 to the next $2k$ vertices. In this case, $ev_h(0) = 4t + 2k + 1$, $ev_h(1) = ev_h(2) = 4t + 2k$.

**Case 3.** $n \equiv 0 \pmod{3}$ and $m \equiv 2 \pmod{3}$.

Let $m = 3k + 2$ and $n = 3t$ where $t, k \in \mathbb{Z}^+$. Assign the label 0 to the $k + 1$ vertices and 2 to the next $2k + 1$ vertices. Here, $ev_h(0) = ev_h(1) = ev_h(2) = 4t + 2k + 1$.

**Case 4.** $n \equiv 1 \pmod{3}$ and $m \equiv 0 \pmod{3}$. 
Let \( m = 3k \) and \( n = 3t + 1 \) where \( t, k \in \mathbb{Z}^+ \). Similar to Case 1. Here, \( ev_h(0) = ev_h(1) = ev_h(2) = 4t + 2k + 1 \).

**Case 5.** \( n \equiv 1 \pmod{3} \) and \( m \equiv 1 \pmod{3} \).

Let \( m = 3k + 1 \) and \( n = 3t + 1 \) where \( t, k \in \mathbb{Z}^+ \). Assign the label 0 to the \( k \) vertices and 2 to the next \( 2k + 1 \) vertices. Here, \( ev_h(0) = 4t + 2k + 1, ev_h(1) = ev_h(2) = 4t + 2k + 2 \).

**Case 6.** \( n \equiv 1 \pmod{3} \) and \( m \equiv 2 \pmod{3} \).

Let \( m = 3k + 2 \) and \( n = 3t + 1 \) where \( t, k \in \mathbb{Z}^+ \). Similar to Case 5. Here, \( ev_h(0) = 4t + 2k + 3, ev_h(1) = ev_h(2) = 4t + 2k + 2 \).

**Case 7.** \( n \equiv 2 \pmod{3} \) and \( m \equiv 0 \pmod{3} \).

Let \( m = 3k \) and \( n = 3t + 2 \) where \( t, k \in \mathbb{Z}^+ \). Similar to Case 1. Here, \( ev_h(0) = 4t + 2k + 3, ev_h(1) = ev_h(2) = 4t + 2k + 2 \).

**Case 8.** \( n \equiv 2 \pmod{3} \) and \( m \equiv 1 \pmod{3} \).

Let \( m = 3k + 1 \) and \( n = 3t + 2 \) where \( t, k \in \mathbb{Z}^+ \). Similar to Case 5. Here, \( ev_h(0) = ev_h(1) = ev_h(2) = 4t + 2k + 3 \).

**Case 9.** \( n \equiv 2 \pmod{3} \) and \( m \equiv 2 \pmod{3} \).

Let \( m = 3k + 2 \) and \( n = 3t + 2 \) where \( t, k \in \mathbb{Z}^+ \). Assign 0 to \( k \) vertices and 2 to \( 2k + 2 \) vertices. Then, \( ev_h(0) = 4t + 2k + 3, ev_h(1) = ev_h(2) = 4t + 2k + 4 \). \( \square \)

**Example 3.2.** A Total Mean Cordial labeling of \( B_{y_6,6} \) is given in figure 4.

![Figure 4](image-url)
Theorem 3.4. The dumbbell graph $Db_n$ is Total Mean Cordial.

Proof. It is clear that $|V(Db_n)| + |E(Db_n)| = 4n + 1$.

Case 1. $n \equiv 0 \pmod{3}$.

For $n = 3$, the figure 5 establish that $Db_3$ is Total Mean Cordial.

\begin{center}
\begin{tikzpicture}
\node[draw,shape=circle,fill=black] (1) at (0,0) {2};
\node[draw,shape=circle,fill=white] (2) at (1,0) {0};
\node[draw,shape=circle,fill=black] (3) at (2,0) {0};
\node[draw,shape=circle,fill=black] (4) at (3,0) {2};
\draw (1) -- (2);
\draw (3) -- (4);
\draw (2) -- (3);
\end{tikzpicture}
\end{center}

\textbf{Figure 5}

Let $n = 3t$ where $t \in \mathbb{Z}^+$ and $t > 1$. Then assign the labels to the vertices of the two cycles as in theorem 2.2. Here, $ev_f(0) = 4t + 1$, $ev_f(1) = ev_f(2) = 4t$.

Case 2. $n \equiv 1 \pmod{3}$.

Let $n = 3t + 1$ where $t \in \mathbb{Z}^+$. Then assign the labels to the vertices of the two cycles as in theorem 2.2. Here, $ev_f(0) = ev_f(1) = 4t + 2$, $ev_f(2) = 4t + 1$.

Case 3. $n \equiv 2 \pmod{3}$.

Let $n = 3t + 1$ where $t \in \mathbb{Z}^+$. Without loss of generality join $u_1$ and $v_1$. Define a function $f : V(Db_n) \to \{0, 1, 2\}$ by

\begin{align*}
    f(u_i) &= 0, \quad 1 \leq i \leq 2t + 2 \\
    f(u_{2t+2+i}) &= 1, \quad 1 \leq i \leq t \\
    f(v_i) &= 2, \quad 1 \leq i \leq 2t + 1 \\
    f(u_{2t+1+i}) &= 1, \quad 1 \leq i \leq t + 1.
\end{align*}

In this case, $ev_f(0) = ev_f(1) = ev_f(2) = 4t + 3$.

Hence Dumbbell graph is Total Mean Cordial. \hfill \Box

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