Longitudinal Space Charge Impedance of Non-Resistive Cylindrical Pipe in the Presence of a Uniform Background of Charged Particles

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Abstract

The longitudinal space charge impedance has been investigated in a smooth perfectly conducting cylindrical beam pipe in the presence of a uniformly distributed background of charged particles. Expressions for the longitudinal space charge impedance, at any point r from the beam axis and at the beam surface r = a, have been obtained. Presence of the background particles introduces a new coupling between the beam and its environment and has been found to increase the longitudinal space charge impedance for low and moderate mode frequencies, while short wavelength modes are unaffected by the presence of the background. Variations of the impedance with the background plasma frequency show a maximum increase of two orders of magnitudes compared to the impedance value in the absence of the background. Numerical examples of the modifications on the longitudinal space charge impedance are shown graphically for an arbitrary choice of beam energies of $\gamma = 1.1$ and $\gamma = 2$, and for a ring of circumference $L = 216$ m, beam pipe radius $b = 0.1$ m and beam radius $a = 0.5$ b.

Keywords: Beam Dynamics, Longitudinal Space Charge, Impedance, Plasma Frequency

Introduction

Behavior of a single beam particle interacting with electromagnetic fields induced inside the beam-pipe and the collective interaction of the whole beam with its surrounding is usually described in terms of coupling impedances [1].

Electrons accumulation in the vacuum chamber in which a positively charged bunch particle beam propagates can deteriorate the vacuum [2, 3] causing interference on the electrodes of beam pickup monitors [4] and can cause beam instabilities [5-14].

Photocemission and secondary emission are known to give rise to a quasistationary electron cloud inside the beam pipe through a beam-induced multipacting process. Rumolo, Ruggiero and Zimmermann [4] investigated the electron-cloud build up and related effects via computer simulation. It has been assumed that macro-particles

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representing photoelectrons are emitted synchronously with the passing proton or positron bunch and are subsequently accelerated in the field of the beam. As they hit the beam pipe, new macro-particles are generated, whose charges are determined by the energy of the incoming particles and by the secondary emission yield of the beam pipe. A quasistationary state of the electron cloud due to space charge was reached and then used as an input parameter for analyzing the electron-cloud driven single-bunch instability [4].

G. Rumolo et. al. [15] also considered the interaction between a low-density electron cloud and a circulating charged particle beam. It is found that particle beam's space charge attracts the cloud, enhancing the cloud density near the beam axis. It has also been shown that the charge enhancement and the image charges associated with the cloud charge and the conducting wall of the accelerator may have important consequences for the dynamics of the beam propagation. Tune shift due to the electron cloud has been obtained analytically and then compared with a numerical model (QUICKPIC) via sample numerical results.

In positron or proton storage rings with many closely spaced bunches, an electron cloud can build up in the vacuum chamber due to photoemission or secondary emission. Ohmi and Zimmermann [16] discussed the possibility of a single-bunch two-stream instability driven by this electron cloud. Depending on the strength of the beam-electron interaction, the chromaticity and the synchrotron oscillation frequency, it is found that this instability either resembles a finite beam breakup or a head-tail instability. Fischer et. al. [17] measured electron cloud densities by observing the coherent tune shift along the bunch train with different bunch spacings and intensities. From the measured coherent tune shifts, electron cloud densities were obtained and compared with densities obtained in electron cloud simulations.

The coupling impedance of straight, uniform beams in a concentric, cylindrical vacuum chamber, whose walls consist of many layers of different materials was treated by Zotter [18]. Zotter and Kheifets [19] derived an expression for the total impedance at the beam surface \( r = a \). S. Kurennoy [20] and J. Wang [21] reviewed the definition of the longitudinal space charge impedance and the corresponding geometry factors for smooth chambers of perfectly conducting walls in the long wavelength approximation. Bisognano [22] has investigated solitary waves in non-relativistic particle beams, where an expression for the ratio of the Fourier transformed potential to density was derived. This ratio is the geometry factor of the longitudinal coupling impedance of a pipe of infinite wall conductivity.

Al-khateeb et. al. [23] calculated both the space charge and the resistive wall impedances for all wavelengths and arbitrary \( \beta \). Expressions for the corresponding generalized and approximate geometry factors have been derived. For non-relativistic particle beams with a finite size, space charge and the resistive wall impedances are found to be of importance for the longitudinal beam dynamics and for the longitudinal beam instability analysis. Experimental determination of the geometry factor for longitudinal perturbations in a space charge dominated beam is found in J. Wang et. al.
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[24]. It was found that the geometry factor obeys the relation \( g = 2 \ln(b/a) \), where \( a \) and \( b \) are the beam and pipe radii, respectively.

The presence of low frequency residual gas in the beam pipe introduces modifications on the space charge impedance (residual contribution). Such a modification will be examined analytically in this paper due to the lack of the quantitative investigation of the contribution of the residual particles to the coupling impedance. The paper is organized as follows: In Sec. II derivation of the electromagnetic fields in a perfectly conducting beam-pipe in the presence of a background of uniformly distributed charged particles will be presented. In Sec. III, expressions for the space charge impedance including the residual part and the corresponding generalized geometry factor will be derived and analyzed. Finally, discussions and conclusions will be presented in Sec IV.

Electromagnetic Fields in a Cylindrical Pipe (emf)

Consider the motion of a beam as rotationally symmetric lamina of particles of radius \( a \) and total charge \( Q \) in a smooth cylindrical pipe of radius \( b \). The beam moves with a constant longitudinal velocity \( \beta c \hat{z} \) along the \( z \) axis through a background of \( n_b \) particles per unit volume, each of mass \( m_0 \) and of charge \( q_b \). The beam current and charge densities \( j_b(r,t) \) and \( \rho_b(r,t) \), respectively, are related as follows,

\[
j_b(r,t) = \rho_b \beta c \hat{z} = \frac{Q}{\pi a^2} \delta(z-\beta c t) \hat{z},
\]

\[
\frac{\partial \rho_b(r,t)}{\partial t} + \nabla \cdot j_b(r,t) = 0
\]

We assume that the background particles are noninteracting and uniformly distributed within the beam pipe such that \( n_b = n_0 \). The bulk motion of the background particles in the electromagnetic fields \( (E, B) \) excited by the beam in the pipe give rise to a net current which, to first order, is approximated by \( j_b = q_b n_0 v_b \). The velocity \( v_b \) is governed by the following equation,

\[
\frac{d v_b(r,t)}{dt} = \frac{q_b}{m_0} \left( E(r,t) + \nu_b(r,t) \times B(r,t) \right)
\]

Let the total current be \( J = j_b + j_b \), the total charge density be \( \rho = \rho_b + q_b n_0 \), and upon using Faraday's and Ampere's laws, the wave equations satisfied by the magnetic \( B \) and electric \( E \) fields in nonconducting linear media with \( \varepsilon = \varepsilon_0 \) and \( \mu = \mu_0 \) (\( \varepsilon_0 \) and \( \mu_0 \) are, respectively, the permittivity and permeability of free space) are,
\[ \nabla^2 B(r,t) - \frac{1}{c^2} \frac{\partial^2 B(r,t)}{\partial t^2} = -\mu_0 \nabla \times J(r,t) \]  
(4)

\[ \nabla^2 E(r,t) - \frac{1}{c^2} \frac{\partial^2 E(r,t)}{\partial t^2} = \mu_0 \frac{\partial M(r,t)}{\partial t} + \frac{\nabla \rho(r,t)}{\varepsilon_0} \]  
(5)

Due to the rotational symmetry, the only non-vanishing field components excited by the beam are \( E_z(r,\omega) \), \( E_r(r,\omega) \) and \( B_\theta(r,\omega) \) \([19, 23]\). According to eq. (3), longitudinal and radial velocity components of the background particles are coupled via the magnetic field \( B(r,t) \). By ignoring the cyclotron motion of the background particles so that they are taken as unmagnetized particles, only electric forces in the longitudinal and radial directions will act on them. This is justified as long as the background response to the beam wake fields is nonrelativistic. Accordingly, the in time Fourier-transformed current and charge densities for \( \omega = k_z v \) are,

\[ J(r, z, \omega) = \frac{Q}{\pi a^2} e^{ik_z z} \hat{z} + \frac{q_0^2 n_0}{m_0 \omega} E(r, z, \omega) \]  
(6)

\[ \rho(r, z, \omega) = \frac{Q}{\pi a^2 \beta c} e^{ik_z z} + 2 \pi q_0 n_0 \delta(\omega) \]  
(7)

Since the time transformed \( \rho_\theta(r,z,\omega) \) and \( J_\theta(r,z,\omega) \) vary with \( z \) as \( e^{ik_z z} \), we assume that the fields \( E \) and \( B \) have the same variation with \( z \) namely, \( E(r, z, \omega) = E(r, \omega) e^{ik_z z} \) and \( B(r, z, \omega) = B(r, \omega) e^{ik_z z} \). Upon Fourier transforming equations 4 and 5 in time, we get,

\[ \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) - k_z^2 \right] B(r, \omega) + \frac{\omega^2}{c^2} B(r, \omega) = \frac{i \omega_0}{c^2} \beta c E(r, \omega) \]  
(8)

\[ \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) - k_z^2 \right] E(r, \omega) + \frac{\omega^2}{c^2} E(r, \omega) = \left( -i \mu_0 \omega \beta c + \frac{i k_z}{\varepsilon_0} \right) \frac{Q}{\pi a^2 \beta c} \hat{z} + \frac{\omega^2}{c^2} E(r, \omega) \]  
(9)

The nonvanishing field components are governed by the following scalar equations,
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\[
\begin{bmatrix}
\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \frac{k_z^2}{\gamma^2} \\
\frac{d}{dr} + \frac{k_z^2}{\gamma^2} \\
\end{bmatrix}
\begin{bmatrix}
E_z(r, \omega) \\
E_t(r, \omega) \\
B_0(r, \omega) \\
\end{bmatrix}
= 
\begin{bmatrix}
\frac{iQk_z}{\pi a^2 \beta c} + \frac{\omega_p^2}{c^2} E_z(r, \omega) \\
\omega_p^2 E_t(r, \omega) \\
\omega_p^2 B_0(r, \omega) \\
\end{bmatrix}
\]

(10)

where the plasma frequency \(\omega_p\) of the background and the relativistic factor \(\gamma\) are defined as follows,

\[
\omega_p^2 = \frac{n_0 q_0^2}{\varepsilon_0 m_0}, \quad \gamma^{-2} = 1 - \beta^2
\]

Introducing the parameter \(\Gamma_0\) such that

\[
\Gamma_0^2 = \frac{k_z^2}{\gamma^2} + \frac{\omega_p^2}{c^2} = \frac{\omega^2 + \omega_p^2 \beta^2}{\gamma^2 \beta^2 c^2}
\]

(11)

The general solution for the \(z\) component of the electric field is

\[
E_z(r, \omega) = \begin{cases} 
A_1 I_0(\Gamma_0 r) + A_2 K_0(\Gamma_0 r) & r > a \\
A_3 I_0(\Gamma_0 r) - i \frac{Q}{\pi a^2 \varepsilon_0 k_z \beta c} & r \leq a
\end{cases}
\]

(12)

where \(I_n\) and \(K_n\) are the modified Bessel functions of first and second kind, respectively, \(A_1, A_2\) and \(A_3\) are integration constants to be determined by the boundary conditions. The relations between the longitudinal electric field component \(E_z(r, z, \omega)\) and the transverse components \(E_t(r, z, \omega)\) and \(B_0(r, z, \omega)\), namely,

\[
\frac{\partial E_z(r, z, \omega)}{\partial r} = i \frac{k_z c}{\beta \gamma^2} B_0(r, z, \omega), \quad E_t(r, z, \omega) = \frac{c}{\beta} B_0(r, z, \omega)
\]

Upon applying the boundary conditions on \(E_z\) and \(B_0\) at the beam surface \(r = a\) and at the pipe inner surface \(r = b\) for a perfectly conducting pipe wall, we get,

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\[ E_z(r, \omega, \omega_p) = \frac{-i Q k_z}{\pi a b \omega r^2 r_0} \left( I_1(\Gamma_0 r) \left[ \frac{K_0(\Gamma_0 r) - K_\beta(\Gamma_0 r)}{I_0(\Gamma_0 r)} \right] + I_0(\Gamma_0 r) \right) \quad \text{if } r > a \]  

(13)

Longitudinal Coupling Impedance

We shall now calculate the longitudinal impedance as a volume integral over the transverse beam charge distribution. The longitudinal impedance of the beam is defined as follows [19, 23],

\[ Z_l(r, \omega, \omega_p) = \frac{1}{Q^2} \int_{V_{\text{beam}}} d^3 \mathbf{x} E_z(r', z, \omega) \cdot \mathbf{J}^*(r', z, \omega) \]

\[ = \frac{1}{Q^2} \int_{V_{\text{beam}}} d^3 \mathbf{x} E_z(r, z, \omega) \mathbf{J}^*(r, z, \omega) \]

\[ = \frac{2 \pi L}{Q \pi a} \int_0^r \int_0^{r'} E_z (r', \omega) E_z^* (r', \omega) r' dr' - \frac{\omega^2 e_0}{\omega} \frac{2 \pi L}{Q^2} \int_0^{r'} E_z (r', \omega) E_z^* (r', \omega) r' dr' \]  

(14)

Using \( \mathbf{J}(r, z, \omega) \) in eq. (6) and the electric field in eq. (13), the impedance at any point \( r \) from the beam axis becomes,

\[ Z_l(r, \omega, \omega_p) = - \frac{i L k_z}{\pi r_0^2 a^2 e_0 \gamma^* \gamma_0} \left( \frac{r^2}{a^2} - \frac{2 \pi}{a} \left[ K_i(\Gamma_0 a) + K_0(\Gamma_0 b) \right] K_i(\Gamma_0 r) \right) + \]

\[ \frac{Lk_z^2}{\omega \pi e_0^* \gamma^* e_0} \left[ \frac{r^2}{a^2} \left( K(\Gamma_0 a) + K_0(\Gamma_0 b) \right) I_0(\Gamma_0 b) \right] \left( \frac{\Delta \Gamma_0 r}{\Gamma_0 b} \right) + \]

\[ \frac{r^2}{a^2} \left[ K_i(\Gamma_0 a) + K_0(\Gamma_0 b) \right] K_i(\Gamma_0 r) \right] \left( K_0^2(\Gamma_0 r) + K_i^2(\Gamma_0 r) \right) \]  

(15)

In the presence of a uniform background, equation (15) gives the beam-pipe coupling impedance at any point \( r \) from the beam axis for arbitrary \( \beta \). The presence of the background of density \( n_0 \) is included in the parameters \( \Gamma_0 \) and \( \omega_p \). For \( n_0 = 0 \) or \( \omega_p = 0 \) and \( r = a \), eq. (15) reduces into the following well known expression for the longitudinal coupling impedance [19, 22, 23].

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\[ Z_{11}^{(0)}(\omega) = -i \frac{L}{\pi a^2 \varepsilon_0 k_x \beta c} \left[ 1 - 2 \left( \frac{K_0(k_x a)}{I_0(k_x a)} \right) \left( \frac{K_0(k_x b)}{I_0(k_x b)} \right) \right] \]  

(16)

Introducing the harmonic number \( n \) such that \( n = k_x R \), the longitudinal impedance in eq. (15) takes the following form at the beam surface \( r = a \),

\[ Z_\eta(\omega, \omega_p) = -i \frac{Z_0}{\gamma^2} \frac{4}{\gamma^2 \beta^2} \frac{1}{\Gamma_0^2 a^2} \left[ 1 - 2 \left( \frac{K_1(\Gamma_0 a)}{I_0(\Gamma_0 a)} \right) \left( \frac{K_1(\Gamma_0 b)}{I_0(\Gamma_0 b)} \right) \right] \]

\[ - i \frac{Z_0}{4 \gamma^2 \beta^2} \frac{1}{\gamma^2 \beta^2} \frac{1}{\Gamma_0^2 a^2} \left( \frac{K_1(\Gamma_0 a)}{I_0(\Gamma_0 b)} - \frac{K_0(\Gamma_0 b)}{I_0(\Gamma_0 b)} \right) \left( \frac{4 I_1(\Gamma_0 a)}{I_0(\Gamma_0 a)} \right) + \]

\[ \left( \frac{K_1(\Gamma_0 a)}{I_0(\Gamma_0 a)} + \frac{K_0(\Gamma_0 b)}{I_0(\Gamma_0 b)} \right) \left( I_0^2(\Gamma_0 a) + I_1^2(\Gamma_0 a) \right) \]

\[ = -i \frac{Z_0}{\gamma^2 \beta} g(a, b, \Gamma_0) = -i \chi_0 g(a, b, \Gamma_0) \]  

(17)

where \( Z_0 = \frac{1}{\varepsilon_0 c} \) is the vacuum impedance, \( \chi_0 = \frac{Z_0}{\gamma^2 \beta} \) and \( g(a, b, \Gamma_0) \) is a generalized geometry factor defined as follows,

\[ g(a, b, \Gamma_0) = \frac{4}{\Gamma_0^2 a^2} \left[ 1 - 2 \left( \frac{K_1(\Gamma_0 a)}{I_0(\Gamma_0 a)} \right) \left( \frac{K_1(\Gamma_0 b)}{I_0(\Gamma_0 b)} \right) \right] + \]

\[ \frac{4 \omega_p^2}{c^2 \Gamma_0^2} \frac{1}{\Gamma_0^2 a^2} \left( \frac{K_1(\Gamma_0 a)}{I_0(\Gamma_0 b)} - \frac{K_0(\Gamma_0 b)}{I_0(\Gamma_0 b)} \right) \left( \frac{4 I_1(\Gamma_0 a)}{I_0(\Gamma_0 a)} \right) + \]

\[ \left( \frac{K_1(\Gamma_0 a)}{I_0(\Gamma_0 b)} + \frac{K_0(\Gamma_0 b)}{I_0(\Gamma_0 b)} \right) \left( I_0^2(\Gamma_0 a) + I_1^2(\Gamma_0 a) \right) \]  

(18)

Equation (18) gives the modified geometry factor of the longitudinal impedance in the presence of the additional coupling between the background particles and the excited longitudinal electric field in the beam-pipe. The residual contribution to the longitudinal impedance is the difference between equations (17) and (16).
To visualize the effect of the presence of the uniform background on the space charge impedance, the rest of this section is devoted to the numerical analysis of the analytical expression of eq. (17). As representative cases, we follow (Al-khateeb et. al. [25]) and we choose the beam energies $\gamma = 1.1$ and $\gamma = 2$. Other beam-pipe parameters are the ring circumference $L = 216$ m, beam-pipe radius $b = 0.1$ m, and beam radius $a = 0.5b$. Fig. 1 and Fig. 2 show plots of eq. (17) for the positive imaginary part of the impedance $Z_i/n$ vs. harmonic number $n = kR$, for different background densities, for $\gamma = 1.1$ and $\gamma = 2.0$ respectively. Fig. 3 shows the plot of the same equation (17) for the positive imaginary part of the impedance $Z_i/n$ vs. $\omega_p/\omega$ for a fixed harmonic number $n = 10$ for the two values $\gamma = 1.1$ and $\gamma = 2.0$.

Discussion and Conclusions

The contribution of a uniformly distributed background of charged particles to the space charge impedance has been investigated analytically. For a particle beam moving in a perfectly conducting beam-pipe, closed form expression for the longitudinal space-charge impedance has been derived. Residual or background impedance can be defined as the difference between the impedance in the presence of the background ($\omega_p \neq 0$) and that in its absence ($\omega_p = 0$), i.e., the difference between equations (17) and (16).

![Graph](image)

**Fig. 1:** The positive imaginary part of the impedance $Z_i/n$ of equation (17) vs. harmonic number $n = kR$ for different background densities and for the parameters $L = 216$ m, $b = 0.1$ m, $a = 0.5b$, and $\gamma = 1.1$
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Fig. 2: The positive imaginary part of the impedance $Z/n$ of equation (17) vs. harmonic number $n = k_{0}R$ for different background densities and for the parameters $L = 216 \text{ m}$, $b = 0.1 \text{ m}$, $a = 0.5b$, and $\gamma = 2.0$.

From the curves of Fig. 1 and Fig. 2, the positive imaginary part of the impedance $Z/n$ decreases systematically with the harmonic number $n$. All curves for the representative energies $\gamma = 1.1$ and $\gamma = 2.0$ tend toward an asymptotic value as the harmonic number $n$ increases, namely, toward the value of $Z/n$ for $\omega_{p} = 0$. Fig. 3 shows that the impedance increases for about two orders of magnitudes as the ratio $\omega_{p}(\omega)$ is increased; for $\gamma = 1.1$ as well as for $\gamma = 2.0$.

Fig. 3: The positive imaginary part of the impedance $Z/n$ of equation (17) vs. $\omega_{p}/\omega$ for the harmonic number $n = 10$ and for the parameters $L = 216 \text{ m}$, $b = 0.1 \text{ m}$, $a = 0.5b$, and $\gamma = 1.1$ (upper curve) and $\gamma = 2.0$ lower curve.
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The observed increase and modification in the impedance curves introduced by the presence of the background particles contribution can be of importance for modeling the longitudinal dynamics of particle beams and for the instability analysis. In addition to the coupling between the beam electromagnetic fields and the beam-pipe eigenmodes, the presence of the charged background introduces an additional coupling between the beam and its environment, which is responsible for the observed increase and modifications.

The illustrative choice of the parameters is somewhat arbitrary and requires a much more detailed investigation in the future. In particular, a deeper look on this problem requires taking into account the perturbation of the background charge density instead of solving the problem by only accounting for the perturbed current resulting from the quiver motion of the background in the field of the beam.

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