Modular Representation Of PSL (2,7) In Characteristic Two

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Abstract

Representing a unimodular group by matrices in characteristic \( p=2 \) is one of the most important research problems in modular representation theory of groups.

The purpose of the paper is to determine all irreducible modular representations of the unimodular group PSL (2,7) over finite fields of characteristic two; and then to investigate all its ordinary irreducible representations over \( \mathbb{F}_q \), the field of two elements.

Keywords: Modular representation, Decomposition matrix, Blocks, Orbits.

1. Introduction

It is a well-known fact in group theory that the two-unimodular dimple groups \( G=\text{PSL}(2,7) \) and \( \text{PSL}(3,2) \) are isomorphic. \( G \) is of order \( 2^3 \cdot 3 \cdot 7 \), and can be presented as follows: (Coxeter, 1972),

\[
G = \langle x, y : x^7 = y^2 = (xy)^3 = (x4y)4 = 1 \rangle
\]

\( G \) has 6 conjugacy classes of elements, and the 6 ordinary irreducible characters of \( G \) are given in the following table:

| \( |g| \) | \( |c(g)| \) |
|---|---|
| \( 2^3 \cdot 3 \cdot 7 \) | \( 2^1 \) | \( 2^2 \) | \( 3 \) | \( 7^+ \) | \( 7^- \) |
| \( \chi_1 \) | 1 | 1 | 1 | 1 | 1 | 1 |
| \( \chi_2 \) | 7 | -1 | -1 | 1 | 0 | 0 |
| \( \chi_3 \) | 6 | 2 | 0 | 0 | -1 | -1 |
| \( \chi_4 \) | 8 | 0 | 0 | -1 | 1 | L |
| \( \chi_5 \) | 3 | -1 | 1 | 0 | Z | Z |
| \( \chi_6 \) | 3 | -1 | 1 | 0 | Z | Z |

\( |g| = \text{order of } g \text{ in } G; \ |c(g)| = \text{order of centralizer of } g \text{ in } G, \text{ and } Z = \frac{-1+i\sqrt{7}}{2} \)

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The ordinary complex irreducible matrix representations of $G$ were all constructed explicitly (Khanfar, 1984).

2. Representations in Characteristic Two

The following established facts and results in modular representation theory of finite groups would be applied without further reference; (Curtis, 1962; Sere, 1977).

1) Let $G$ be a finite group and $p$ a rational prime number. Then an element $g \in G$ is $p$-regular if its order is relatively prime to $p$, and $p$-singular if its order is a power of $p$.

Since all conjugate elements are of the same order, we speak of the $p$-regular conjugacy classes of $G$.

2) The number of irreducible modular representations of $G$ in a field of characteristic $p$ is equal to the number of $p$-regular conjugacy classes of $G$.

3) If $\varnothing$ is an irreducible ordinary representation of $G$ and, if $p^m$ is the highest power of $p$ dividing the order of $G$ and the degree of $\varnothing$, then $\varnothing$ remains irreducible as a modular representation of $G$ in characteristic $p$.

Now $G = PSL(3,2)$ has four $2$-regular conjugacy classes of elements, and hence four irreducible modular representation in characteristic $2$. The ordinary irreducible complex representations of degree $1$ and $8$ remain irreducible as modular representations of $G$ in characteristic $2$. Following (Curtis, 1962) in distributing the irreducible representations into blocks, we find that the $2$-blocks of ordinary irreducible representations of $G$ are.

$B_1 = \{1,3,3,6,7\}$ of defect $3$,

$B_2 = \{8\}$ of defect $0$;

Where representations in blocks are indicated by their degrees. Thus the two unknown irreducible $2$-modular representations of $G$ are in $B_1$.

In the ordinary $3$-dimensional representation of $G$, the element of order $7$ is represented by a diagonal matrix with distinct $7^{th}$ roots of unity as diagonal entries, (Khanfar, 1984).

$$X^* = \begin{bmatrix} \omega & \omega^2 & \omega^4 \\ \omega & \omega^2 & \omega^4 \\ \omega & \omega^2 & \omega^4 \end{bmatrix} \quad \text{And} \quad X^* = \begin{bmatrix} \omega^3 & \omega^5 & \omega^6 \\ \omega^3 & \omega^5 & \omega^6 \\ \omega^3 & \omega^5 & \omega^6 \end{bmatrix}$$

Since an element of order $7$ in $G$ is $2$-regular, the modular character coincides with the ordinary character on an element of order $7$. Hence a $3$-dimensional representation of $G$ cannot be reducible in characteristic $2$.  

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Since the eigenvalues of $X^+$ are distinct from those of $X^-$, the two 3-dimensional representation of $G$ are also inequivalent in characteristic 2. Thus the irreducible modular degrees in characteristic 2 are 1, 3, 3 and 8.

Using the character table of $G$, we find that the only possibility for the 6-dimensional character is to break up into 3 and 3; and the only possibility for the 7-dimensional character is to break up into 1, 3 and 3. Thus the decomposition matrix of $G$ can now be constructed as follows:

<table>
<thead>
<tr>
<th>Mod.deg.</th>
<th>1</th>
<th>3</th>
<th>3</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ord.deg</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

The following two sections investigate the 6-dimensional and the 7-dimensional representations of $G$ over $F_2$. We shall find that the irreducible 3-dimensional modular representations of $G$ can be written in $F_2$.

3. The 6-dimensional $G$ over $F_2$

The ordinary 6-dimensional representation of $G$ is constructed over $F_3$ in (Khanfar, 1984), and found to be

$$X = \begin{pmatrix}
-1 & 1 & \cdots & \cdots \\
-1 & 1 & \cdots & \cdots \\
-1 & \cdots & 1 & \cdots \\
-1 & \cdots & 1 & \cdots \\
-1 & \cdots & \cdots & 1 \\
-1 & \cdots & \cdots & \cdots 
\end{pmatrix}$$

$$Y = \begin{pmatrix}
\cdots & 1 & \cdots & \cdots \\
\cdots & 1 & \cdots & \cdots \\
\cdots & \cdots & 1 & \cdots \\
\cdots & \cdots & \cdots & 1 \\
\cdots & \cdots & \cdots & \cdots 
\end{pmatrix}$$

Let $e_i$, $i = 1, \ldots, 6$ be the unit vectors in 6 dimensions. As a 6-dimensional group over $F_2$, $G$ permutes the vectors $e_i$, $\sum e_i$, among themselves. We consider the integral lattice $L$ generated by orbit

$$L = \langle e_i, \sum e_i \rangle$$

Then $L/2L$ is a ZG-module; in fact a vector space $V_6(2)$. We investigate the action of $G$ on this space. An element of order 7 in $G$ does not fix any non-zero vector; while an element of order 3 fixes a 2-dimensional sub space. An involution in $G$ fixes a 4-dimensional subspace, and an element of order 4 fixes a 2-dimensional
subspace. Using the orbits of the normalizers of the various elements in $G$ on the fixes subspaces, we determine the stabilizers of vectors and the orbits of $G$ in $V_6(2)$. The table below gives a representative and length for each orbit:

<table>
<thead>
<tr>
<th>Representative</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_1$</td>
<td>$e_1$</td>
</tr>
<tr>
<td>$L_2$</td>
<td>$e_1 + e_2 + e_3 + e_6$</td>
</tr>
<tr>
<td>$L_3$</td>
<td>$e_1 + e_2$</td>
</tr>
<tr>
<td>$L_4$</td>
<td>$e_1 + e_3 + e_6$</td>
</tr>
</tbody>
</table>

The trivial orbit with $L_2$ from a unique $G$- invariant subspace:

$$U_3(2) = \left\{ e_1 + e_2 + e_3 + e_6, e_2 + e_4 + e_5 + e_6, e_3 + e_4 + e_6 \right\}$$

Extending this basis of $U_3(2)$ to a basis for $V_6(2)$, we obtain

$$X = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}, \quad y = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}$$

The two 3-dimensional representations of $G$ over $F_2$ thus obtained, are irreducible because $U_3(2)$ has no 1-dimensional $G$-invariant subspace. Moreover, it is easily seen that these representations are inequivalent over $F_2$.

4. The 7-dimensional $G$ over $F_2$

the 7-dimensional representation of $G$ is over $F_3$ (Khanfar, 1984), and it was found to be as follows:

$$X = \begin{bmatrix}
1 & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . \\
. & . & 1 & . & . & . & . \\
. & . & . & 1 & . & . & . \\
. & . & . & . & 1 & . & . \\
. & . & . & . & . & 1 & . \\
1 & . & . & . & . & . & 1
\end{bmatrix}, \quad y = \begin{bmatrix}
-1 & . & . & . & . & . & . \\
-1 & . & . & . & . & . & 1 \\
-1 & . & . & . & 1 & . & . \\
-1 & . & 1 & . & . & . & . \\
-1 & . & . & 1 & . & . & . \\
-1 & . & . & 1 & . & . & . \\
\end{bmatrix}$$

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If \( e_i, i=1,\ldots, 7 \) denote the unit vectors 7-dimensions, then the 7-dimensional \( G \) over \( F_2 \) permutes the 8 vectors \( e_1, \ldots, e_7 \) among themselves.

Considering the integral lattice \( L \) generated by this orbit, we have \( L/2L \) as a 7-dimensional space over \( F_2 \). The action of \( G \) on this space is as follows:

The elements of \( G \) of orders 7, 3, 2 and 4 fix respectively 1-dimensional, 3-dimensional, 4-dimensional and 2-dimensional subspaces. Using the orbits in these fixed subspaces under the actions of the normalizers of the various elements in \( G \), we determine the stabilizers of vectors and the orbits of \( G \) in the 7-dimensional space over \( F_2 \). The following table gives a representative and the length of each orbit:

**Table 3.**

<table>
<thead>
<tr>
<th>Representative</th>
<th>length</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L_1 )</td>
<td>( e_1 + e_2 + e_3 + e_5 )</td>
</tr>
<tr>
<td>( L_2 )</td>
<td>( e_1 + e_2 + e_3 + e_6 )</td>
</tr>
<tr>
<td>( L_3 )</td>
<td>( e_1 )</td>
</tr>
<tr>
<td>( L_4 )</td>
<td>( e_1 + e_2 + e_3 + e_4 )</td>
</tr>
<tr>
<td>( L_5 )</td>
<td>( e_1 + e_2 )</td>
</tr>
<tr>
<td>( L_6 )</td>
<td>( e_1 + e_2 + e_3 )</td>
</tr>
</tbody>
</table>

The union of the orbits (0) \( L_1, L_2, L_4 \) and \( L_5 \) forms a unique \( G \)-invariant 6-dimensional subspace \( U \). In turn, \( U \) is the direct sum of two \( G \)-invariant 3-dimensional subspaces \( V_1 \) and \( V_2 \), where \( V_1 \) is the union of the orbit (0) with \( L_i, i=1,2 \).

Adapting bases of these two subspaces to a basis for the 7-dimensional space over \( F_2 \), we obtain:

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}
\]
Thus, again we obtain two 3-dimensional representations of \( G \) over \( F_2 \), these representations are irreducible since neither \( V_1 \) nor \( V_2 \) contains a 1-dimensional \( G \)-invariant subspace. It is easily seen that these representations are also inequivalent over \( F_2 \); but are equivalent to the representations obtained in the previous section.

Finally, we remark here that the 8-dimensional representation of \( G = \text{PSL}(3,2) \) is monomial and was constructed by (Khanfar, 1984) over the complex field with entries the cube roots of unity. This representation can also be written over \( F_2 \); but the verification of this claim here tends to be rather lengthy to be included in this paper. However, (Khanfar, 1986) has obtained that representation over \( F_2 \) as a composition factor of a 28-dimensional representation in \( F_2 \) of a monomial subgroup of the Rudvalis simple group. It was found to be:

\[
X = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

\[
, \ y = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

The author has certified that the generator relations of \( G \) are satisfied by this representation over \( F_2 \) and that it is, indeed, irreducible.

Acknowledgments

The author is very grateful to those graduate students at Yarmouk University who helped in hand computations. Gratitude is due also in advance to the editor and referees for their consideration of this work.

"تمثيل الزمرة (7,2) في المميز اثنين"  
محمد خنفر

العنوان

تمثيل زمر أحادية بالمجموعات في حقل متميزة من أهم مسائل البحث في نظرية تمثيل الزمر. يهدف هذا البحث إلى تعيين كل التمثيلات المختلفة وغير المتكافئة للزمرة الأحادية PSL(2,7) في الحقل المتميزة ذات الميزة 2. ومن ثم فحص كل التمثيلات العادية للزمرة PSL(2,7) في حقل F2 من الصغرى إلى الأكبر. 

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References


