Abstract. Characterizations and properties of $\mathcal{I}^*g$-closed sets and $\mathcal{I}^*g$-open sets are given. A characterization of normal spaces is given in terms of $\mathcal{I}^*g$-open sets. Also, it is established that an $\mathcal{I}^*g$-closed subset of an $\mathcal{I}$-compact space is $\mathcal{I}$-compact.

1. Introduction and preliminaries

An ideal $\mathcal{I}$ on a topological space $(X, \tau)$ is a nonempty collection of subsets of $X$ which satisfies (i) $A \in \mathcal{I}$ and $B \subset A \Rightarrow B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$. Given a topological space $(X, \tau)$ with an ideal $\mathcal{I}$ on $X$ and if $\varphi(X)$ is the set of all subsets of $X$, a set operator $(.)^* : \varphi(X) \rightarrow \varphi(X)$, called a local function [8] of $A$ with respect to $\tau$ and $\mathcal{I}$ is defined as follows: for $A \subset X$, $A^*(\mathcal{I}, \tau) = \{x \in X \mid U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau \mid x \in U\}$. We will make use of the basic facts about the local functions [[7], Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $\text{cl}^*(.)$ for a topology $\tau^*(\mathcal{I}, \tau)$, called the $\star$-topology, finer than $\tau$ is defined by $\text{cl}^*(A) = \text{A} \cup A^*(\mathcal{I}, \tau)$ [16]. When there is no chance for confusion, we will simply write $A^*$ for $A^*(\mathcal{I}, \tau)$ and $\tau^*$ for $\tau^*(\mathcal{I}, \tau)$.

If $\mathcal{I}$ is an ideal on $X$, then $(X, \tau, \mathcal{I})$ is called an ideal space. $\mathcal{N}$ is the ideal of all nowhere dense subsets in $(X, \tau)$. A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is $\star$-closed [7]

2000 Mathematics Subject Classification. 54A05, Secondary 54D15, 54D30.

Key words and phrases. *g-closed set, $\mathcal{I}^*g$-closed set and $\mathcal{I}$-compact space.

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Received: Nov 15, 2011 Accepted : Oct. 10 , 2012 .
(resp. $\star$-dense in itself [5]) if $A^* \subseteq A$ (resp. $A \subseteq A^*$). A subset $A$ of an ideal space $(X, \tau, I)$ is $I_\gamma$-closed [2] if $A^* \subseteq U$ whenever $A \subseteq U$ and $U$ is open.

By a space, we always mean a topological space $(X, \tau)$ with no separation properties assumed. If $A \subseteq X$, $\text{cl}(A)$ and $\text{int}(A)$ will, respectively, denote the closure and interior of $A$ in $(X, \tau)$ and $\text{int}^*(A)$ will denote the interior of $A$ in $(X, \tau^*)$.

A subset $A$ of a space $(X, \tau)$ is an $\alpha$-open [14] (resp. semi-open [9], preopen [11]) set if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$ (resp. $A \subseteq \text{cl}(\text{int}(A))$, $A \subseteq \text{int}(\text{cl}(A))$).

The family of all $\alpha$-open sets in $(X, \tau)$, denoted by $\tau_\alpha$, is a topology on $X$ finer than $\tau$. The closure of $A$ in $(X, \tau_\alpha)$ is denoted by $\text{cl}_\alpha(A)$.

**Definition 1.1.** A subset $A$ of a space $(X, \tau)$ is said to be

1. $g$-closed [10] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open,
2. $\hat{g}$-closed [17] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open,
3. $\hat{g}$-open [17] if its complement is $\hat{g}$-closed,
4. $^*g$-closed [6] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\hat{g}$-open.

The family of all $\hat{g}$-open sets in $(X, \tau)$ is a topology on $X$. The $\hat{g}$-closure [17] of a subset $A$ of $X$, denoted by $\hat{g}\text{cl}(A)$, is defined to be the intersection of all $\hat{g}$-closed sets containing $A$.

**Definition 1.2.** An ideal $I$ is said to be

1. codense [3] or $\tau$-boundary [13] if $\tau \cap I = \{\emptyset\}$,
2. completely codense [3] if $\text{PO}(X) \cap I = \{\emptyset\}$, where $\text{PO}(X)$ is the family of all preopen sets in $(X, \tau)$.

**Lemma 1.1.** Every completely codense ideal is codense but not the converse [3].

The following Lemmas will be useful in the sequel.
Lemma 1.2. Let \((X,\tau,\mathcal{I})\) be an ideal space and \(A \subseteq X\). If \(A \subseteq A^*\), then \(A^* = cl(A^*) = cl(A) = cl^* (A) \) \([15], \text{Theorem 5}\).

Lemma 1.3. Let \((X,\tau,\mathcal{I})\) be an ideal space. Then \(\mathcal{I}\) is codense if and only if \(G \subseteq G^*\) for every semi-open set \(G\) in \(X\) \([15], \text{Theorem 3}\).

Lemma 1.4. Let \((X,\tau,\mathcal{I})\) be an ideal space. If \(\mathcal{I}\) is completely codense, then \(\tau^* \subseteq \tau^0\) \([15], \text{Theorem 6}\).

Remark 1. If \((X,\tau)\) is a topological space, then every closed set is \(\hat{g}\)-closed but not conversely \([1], \text{Theorem 2.3}\).

Lemma 1.5. If \((X,\tau,\mathcal{I})\) is a \(T_1\) ideal space and \(A\) is an \(I_g\)-closed set, then \(A\) is a \(\ast\)-closed set \([12], \text{Corollary 2.2}\).

Lemma 1.6. Every \(g\)-closed set is \(I_g\)-closed but not conversely \([2], \text{Theorem 2.1}\).

2. \(I_{sg}\)-closed sets

Definition 2.1. A subset \(A\) of an ideal space \((X,\tau,\mathcal{I})\) is said to be

(1) \(I_{sg}\)-closed if \(A^* \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\hat{g}\)-open,

(2) \(I_{sg}\)-open if its complement is \(I_{sg}\)-closed.

Theorem 2.1. If \((X,\tau,\mathcal{I})\) is any ideal space, then every \(I_{sg}\)-closed set is \(I_g\)-closed but not conversely.

Example 2.1. Let \(X = \{a,b,c\}\), \(\tau = \{\emptyset, X, \{c\}, \{a,b\}\}\) and \(\mathcal{I} = \{\emptyset, \{a\}\}\). Then \(I_{sg}\)-closed sets are \(\emptyset, X, \{a\}, \{c\}, \{a,b\}\), \(\{a,c\}\) and \(I_g\)-closed sets are power set of \(X\). It is clear that \(\{b\}\) is \(I_g\)-closed but it is not \(I_{sg}\)-closed.

The following theorem gives characterizations of \(I_{sg}\)-closed sets.

Theorem 2.2. If \((X,\tau,\mathcal{I})\) is any ideal space and \(A \subseteq X\), then the following are equivalent.
(1) A is $I_{sg}$-closed,

(2) $cl^*(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\hat{g}$-open in $X$,

(3) For all $x \in cl^*(A)$, $\hat{g}cl(\{x\}) \cap A \neq \emptyset$.

(4) $cl^*(A) - A$ contains no nonempty $\hat{g}$-closed set,

(5) $A^* - A$ contains no nonempty $\hat{g}$-closed set.

Proof. (1)⇒(2) If $A$ is $I_{sg}$-closed, then $A^* \subseteq U$ whenever $A \subseteq U$ and $U$ is $\hat{g}$-open in $X$ and so $cl^*(A) = A \cup A^* \subseteq U$ whenever $A \subseteq U$ and $U$ is $\hat{g}$-open in $X$. This proves (2).

(2)⇒(3) Suppose $x \in cl^*(A)$. If $\hat{g}cl(\{x\}) \cap A = \emptyset$, then $A \subseteq X - \hat{g}cl(\{x\})$. By (2), $cl^*(A) \subseteq X - \hat{g}cl(\{x\})$, a contradiction, since $x \in cl^*(A)$.

(3)⇒(4) Suppose $F \subseteq cl^*(A) - A$, $F$ is $\hat{g}$-closed and $x \in F$. Since $F \subseteq X - A$ and $F$ is $\hat{g}$-closed, then $A \subseteq X - F$ and $F$ is $\hat{g}$-closed, $\hat{g}cl(\{x\}) \cap A \neq \emptyset$. Since $x \in cl^*(A)$ by (3), $\hat{g}cl(\{x\}) \cap A \neq \emptyset$. Therefore $cl^*(A) - A$ contains no nonempty $\hat{g}$-closed set.

(4)⇒(5) Since $cl^*(A) - A = (A \cup A^*) - A = (A \cup A^*) \cap A^c = (A^c \cap A^c) \cup (A^c \cap A^c) = A^c \cap A^c = A^* - A$. Therefore $A^* - A$ contains no nonempty $\hat{g}$-closed set.

(5)⇒(1) Let $A \subseteq U$ where $U$ is $\hat{g}$-open. Therefore $X - U \subseteq X - A$ and so $A^* \cap (X - U) \subseteq A^* \cap (X - A) = A^* - A$. Therefore $A^* \cap (X - U) \subseteq A^* - A$. Since $A^*$ is always closed set, so $A^*$ is $\hat{g}$-closed set and so $A^* \cap (X - U)$ is a $\hat{g}$-closed set contained in $A^* - A$. Therefore $A^* \cap (X - U) = \emptyset$ and hence $A^* \subseteq U$. Therefore $A$ is $I_{sg}$-closed.

**Theorem 2.3.** Every $\star$-closed set is $I_{sg}$-closed but not conversely.

Proof. Let $A$ be a $\star$-closed, then $A^* \subseteq A$. Let $A \subseteq U$ where $U$ is $\hat{g}$-open. Hence $A^* \subseteq U$ whenever $A \subseteq U$ and $U$ is $\hat{g}$-open. Therefore $A$ is $I_{sg}$-closed.

**Example 2.2.** Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a,b\}\}$ and $I = \{\emptyset\}$. Then $I_{sg}$-closed sets are $\emptyset, X, \{c\}, \{a,c\}, \{b,c\}$ and $\star$-closed sets are $\emptyset, X, \{c\}$. It is clear that $\{a,c\}$ is an $I_{sg}$-closed set but it is not $\star$-closed.

**Theorem 2.4.** Let $(X, \tau, I)$ be an ideal space. For every $A \in I$, $A$ is $I_{sg}$-closed.
Proof. Let \( A \subseteq U \) where \( U \) is \( \hat{g} \)-open set. Since \( A^* = \emptyset \) for every \( A \in I \), then \( \text{cl}^*(A) = A \cup A^* = A \subseteq U \). Therefore, by Theorem 2.2, \( A \) is \( \mathcal{I}_{sg} \)-closed.

**Theorem 2.5.** If \((X, \tau, \mathcal{I})\) is an ideal space, then \( A^* \) is always \( \mathcal{I}_{sg} \)-closed for every subset \( A \) of \( X \).

**Proof.** Let \( A^* \subseteq U \) where \( U \) is \( \hat{g} \)-open. Since \( (A^*)^* \subseteq A^* \) [7], we have \( (A^*)^* \subseteq U \) whenever \( A^* \subseteq U \) and \( U \) is \( \hat{g} \)-open. Hence \( A^* \) is \( \mathcal{I}_{sg} \)-closed.

**Theorem 2.6.** Let \((X, \tau, \mathcal{I})\) be an ideal space. Then every \( \mathcal{I}_{sg} \)-closed, \( \hat{g} \)-open set is \( \star \)-closed set.

**Proof.** Since \( A \) is \( \mathcal{I}_{sg} \)-closed and \( \hat{g} \)-open. Then \( A^* \subseteq A \) whenever \( A \subseteq A \) and \( A \) is \( \hat{g} \)-open. Hence \( A \) is \( \star \)-closed.

**Corollary 2.1.** If \((X, \tau, \mathcal{I})\) is a \( T_I \) ideal space and \( A \) is an \( \mathcal{I}_{sg} \)-closed set, then \( A \) is \( \star \)-closed set.

**Corollary 2.2.** Let \((X, \tau, \mathcal{I})\) be an ideal space and \( A \) be an \( \mathcal{I}_{sg} \)-closed set. Then the following are equivalent.

1. \( A \) is a \( \star \)-closed set,
2. \( \text{cl}^*(A) - A \) is a \( \hat{g} \)-closed set,
3. \( A^* - A \) is a \( \hat{g} \)-closed set.

**Proof.** (1) \( \Rightarrow \) (2) If \( A \) is \( \star \)-closed, then \( A^* \subseteq A \) and so \( \text{cl}^*(A) - A = (A \cup A^*) - A = \emptyset \). Hence \( \text{cl}^*(A) - A \) is \( \hat{g} \)-closed set.

(2) \( \Rightarrow \) (3) Since \( \text{cl}^*(A) - A = A^* - A \) and so \( A^* - A \) is \( \hat{g} \)-closed set.

(3) \( \Rightarrow \) (1) If \( A^* - A \) is a \( \hat{g} \)-closed set, since \( A \) is \( \mathcal{I}_{sg} \)-closed set, by Theorem 2.2, \( A^* - A = \emptyset \) and so \( A \) is \( \star \)-closed.

**Theorem 2.7.** Let \((X, \tau, \mathcal{I})\) be an ideal space. Then every \( *g \)-closed set is an \( \mathcal{I}_{sg} \)-closed set but not conversely.
Proof. Let $A$ be a $^g$-closed set. Then $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $^g$-open. We have $\text{cl}^*(A) \subseteq \text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $^g$-open. Hence $A$ is $I^g$-closed.

Example 2.3. Let $X=\{a,b,c\}$, $\tau=\{\emptyset,X,\{c\},\{a,b\}\}$ and $I=\{\emptyset,\{a\}\}$. Then $I^g$-closed sets are $\emptyset, X, \{a\}, \{c\}, \{a,b\}$ and $^g$-closed sets are $\emptyset, X, \{c\}, \{a,b\}$. It is clear that $\{a\}$ is $I^g$-closed set but it is not $^g$-closed.

Theorem 2.8. If $(X,\tau,I)$ is an ideal space and $A$ is a $*$-dense in itself, $I^g$-closed subset of $X$, then $A$ is $^g$-closed.

Proof. Suppose $A$ is a $*$-dense in itself, $I^g$-closed subset of $X$. Let $A \subseteq U$ where $U$ is $^g$-open. Then by Theorem 2.2 (2), $\text{cl}^*(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $^g$-open. Since $A$ is $*$-dense in itself, by Lemma 1.2, $\text{cl}(A)=\text{cl}^*(A)$. Therefore $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $^g$-open. Hence $A$ is $^g$-closed.

Corollary 2.3. If $(X,\tau,I)$ is any ideal space where $I=\{\emptyset\}$, then $A$ is $I^g$-closed if and only if $A$ is $^g$-closed.

Proof. From the fact that for $I=\{\emptyset\}$, $A^*=\text{cl}(A) \supseteq A$. Therefore $A$ is $*$-dense in itself. Since $A$ is $I^g$-closed, by Theorem 2.8, $A$ is $^g$-closed.

Conversely, by Theorem 2.7, every $^g$-closed set is $I^g$-closed set.

Corollary 2.4. If $(X,\tau,I)$ is any ideal space where $I$ is codense and $A$ is a semi-open, $I^g$-closed subset of $X$, then $A$ is $^g$-closed.

Proof. By Lemma 1.3, $A$ is $*$-dense in itself. By Theorem 2.8, $A$ is $^g$-closed.

Example 2.4. Let $X=\{a,b,c\}$, $\tau=\{\emptyset,X,\{c\},\{a,b\}\}$ and $I=\{\emptyset,\{a\}\}$. Then $g$-closed sets are power set of $X$ and $I^g$-closed sets are $\emptyset,X,\{a\},\{c\},\{a,b\},\{a,c\}$. It is clear that $\{b\}$ is $g$-closed set but it is not $I^g$-closed.
Example 2.5. Let \( X=\{a,b,c\} \), \( \tau=\{\emptyset,X,\{a,b\}\} \) and \( \mathcal{I}=\{\emptyset,\{a\}\} \). Then \( g \)-closed sets are \( \emptyset,X,\{c\},\{a,c\},\{b,c\} \) and \( \mathcal{I}^*_{g} \)-closed sets are \( \emptyset,X,\{a\},\{c\},\{a,c\},\{b,c\} \). It is clear that \( \{a\} \) is \( \mathcal{I}^*_{g} \)-closed set but it is not \( g \)-closed.

Remark 2. By Example 2.4 and Example 2.5, \( g \)-closed sets and \( \mathcal{I}^*_{g} \)-closed sets are independent.

Remark 3. We have the following implications for the subsets stated above.

\[
\text{closed} \quad \xrightarrow{\hat{g}\text{-closed}} \quad \text{\( \mathcal{I}^*_{g} \)-closed} \quad \xrightarrow{g\text{-closed}} \quad \text{\( \mathcal{I}^* \)-closed}
\]

Theorem 2.9. Let \((X,\tau,\mathcal{I})\) be an ideal space and \( A \subseteq X \). Then \( A \) is \( \mathcal{I}^*_{g} \)-closed if and only if \( A=F-N \) where \( F \) is \( \ast \)-closed and \( N \) contains no nonempty \( \hat{g} \)-closed set.

Proof. If \( A \) is \( \mathcal{I}^*_{g} \)-closed, then by Theorem 2.2 (5), \( N=A^*-A \) contains no nonempty \( \hat{g} \)-closed set. If \( F=\text{cl}^*(A) \), then \( F \) is \( \ast \)-closed such that \( F-N=(A\cup A^*)-(A^*-A)=(A\cup A^*)\cap (A\cap A^*)^c=(A\cup A^*)\cap (A\cap (A^*)^c)=A \).

Conversely, suppose \( A=F-N \) where \( F \) is \( \ast \)-closed and \( N \) contains no nonempty \( \hat{g} \)-closed set. Let \( U \) be an \( \hat{g} \)-open set such that \( A \subseteq U \). Then \( F-N \subseteq U \) which implies that \( F \cap (X-U) \subseteq N \). Now \( A \subseteq F \) and \( F \subseteq F^* \) then \( A^* \subseteq F^* \) and so \( A^* \cap (X-U) \subseteq F^* \cap (X-U) \subseteq F \cap (X-U) \subseteq N \). By hypothesis, since \( A^* \cap (X-U) \) is \( \hat{g} \)-closed, \( A^* \cap (X-U)=\emptyset \) and so \( A^* \subseteq U \). Hence \( A \) is \( \mathcal{I}^*_{g} \)-closed.

Theorem 2.10. Let \((X,\tau,\mathcal{I})\) be an ideal space and \( A \subseteq X \). If \( A \subseteq B \subseteq A^* \), then \( A^*=B^* \) and \( B \) is \( \ast \)-dense in itself.
Proof. Since \( A \subseteq B \), then \( A^* \subseteq B^* \) and since \( B \subseteq A^* \), then \( B^* \subseteq (A^*)^* \subseteq A^* \). Therefore \( A^* = B^* \) and \( B \subseteq A^* \subseteq B^* \). Hence proved.

**Theorem 2.11.** Let \((X, \tau, I)\) be an ideal space. If \( A \) and \( B \) are subsets of \( X \) such that \( A \subseteq B \subseteq \text{cl}^* (A) \) and \( A \) is \( I^* g \)-closed, then \( B \) is \( I^* g \)-closed.

**Proof.** Since \( A \) is \( I^* g \)-closed, then by Theorem 2.2 (5), \( \text{cl}^* (A) - A \) contains no nonempty \( \hat{g} \)-closed set. Since \( \text{cl}^* (B) - B \subseteq \text{cl}^* (A) - A \) and so \( \text{cl}^* (B) - B \) contains no nonempty \( \hat{g} \)-closed set. Hence \( B \) is \( I^* g \)-closed.

**Corollary 2.5.** Let \((X, \tau, I)\) be an ideal space. If \( A \) and \( B \) are subsets of \( X \) such that \( A \subseteq B \subseteq A^* \) and \( A \) is \( I^* g \)-closed, then \( A \) and \( B \) are \( *g \)-closed sets.

**Proof.** Let \( A \) and \( B \) be subsets of \( X \) such that \( A \subseteq B \subseteq A^* \) which implies that \( A \subseteq B \subseteq A^* \subseteq \text{cl}^* (A) \) and \( A \) is \( I^* g \)-closed. By Theorem 2.11, \( B \) is \( I^* g \)-closed. Since \( A \subseteq B \subseteq A^* \), then \( A^* = B^* \) and so \( A \) and \( B \) are \( * \)-dense in itself. By Theorem 2.8, \( A \) and \( B \) are \( *g \)-closed.

The following theorem gives a characterization of \( I^* g \)-open sets.

**Theorem 2.12.** Let \((X, \tau, I)\) be an ideal space and \( A \subseteq X \). Then \( A \) is \( I^* g \)-open if and only if \( F \subseteq \text{int}^* (A) \) whenever \( F \) is \( \hat{g} \)-closed and \( F \subseteq A \).

**Proof.** Suppose \( A \) is \( I^* g \)-open. If \( F \) is \( \hat{g} \)-closed and \( F \subseteq A \), then \( X - A \subseteq X - F \) and so \( \text{cl}^* (X - A) \subseteq X - F \) by Theorem 2.2 (2). Therefore \( F \subseteq X - \text{cl}^* (X - A) = \text{int}^* (A) \). Hence \( F \subseteq \text{int}^* (A) \).

Conversely, suppose the condition holds. Let \( U \) be a \( \hat{g} \)-open set such that \( X - A \subseteq U \). Then \( X - U \subseteq A \) and so \( X - U \subseteq \text{int}^* (A) \). Therefore \( \text{cl}^* (X - A) \subseteq U \). By Theorem 2.2 (2), \( X - A \) is \( I^* g \)-closed. Hence \( A \) is \( I^* g \)-open.

**Corollary 2.6.** Let \((X, \tau, I)\) be an ideal space and \( A \subseteq X \). If \( A \) is \( I^* g \)-open, then \( F \subseteq \text{int}^* (A) \) whenever \( F \) is closed and \( F \subseteq A \).
The following theorem gives a property of $I_g$-closed.

**Theorem 2.13.** Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subseteq X$. If $A$ is $I_g$-open and $\text{int}^*(A) \subseteq B \subseteq A$, then $B$ is $I_g$-open.

**Proof.** Since $A$ is $I_g$-open, then $X - A$ is $I_g$-closed. By Theorem 2.2 (4), $\text{cl}^*(X-A) - (X-A)$ contains no nonempty $\hat{g}$-closed set. Since $\text{int}^*(A) \subseteq \text{int}^*(B)$ which implies that $\text{cl}^*(X-B) \subseteq \text{cl}^*(X-A)$ and so $\text{cl}^*(X-B) - (X-B) \subseteq \text{cl}^*(X-A) - (X-A)$. Hence $B$ is $I_g$-open.

The following theorem gives a characterization of $I_g$-closed sets in terms of $I_g$-open sets.

**Theorem 2.14.** Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subseteq X$. Then the following are equivalent.

1. $A$ is $I_g$-closed,
2. $A \cup (X - A^*)$ is $I_g$-closed,
3. $A^* - A$ is $I_g$-open.

**Proof.** (1)⇒(2) Suppose $A$ is $I_g$-closed. If $U$ is any $\hat{g}$-open set such that $A \cup (X - A^*) \subseteq U$, then $X - U \subseteq X - (A \cup (X - A^*)) = X \cap (A \cup (A^*)^c) = A^c \cap A^c = A^* - A$. Since $A$ is $I_g$-closed, by Theorem 2.2 (5), it follows that $X - U = \emptyset$ and so $X = U$. Therefore $A \cup (X - A^*) \subseteq U$ which implies that $A \cup (X - A^*) \subseteq X$ and so $(A \cup (X - A^*))^* \subseteq X^* \subseteq X = U$. Hence $A \cup (X - A^*)$ is $I_g$-closed.

(2)⇒(1) Suppose $A \cup (X - A^*)$ is $I_g$-closed. If $F$ is any $\hat{g}$-closed set such that $F \subseteq A^* - A$, then $F \subseteq A^*$ and $F \subseteq X \setminus A$ which implies that $X - A \subseteq X - F$ and $A \subseteq X - F$. Therefore $A \cup (X - A^*) \subseteq A \cup (X - F) = X - F$ and $X - F$ is $\hat{g}$-open. Since $(A \cup (X - A^*))^* \subseteq X - F$ which implies that $A^* \cup (X - A^*)^* \subseteq X - F$ and so $A^* \subseteq X - F$ which implies that $F \subseteq X - A^*$. Since $F \subseteq A^*$, it follows that $F = \emptyset$. Hence $A$ is $I_g$-closed.
(2) ⇔ (3) Since \( X-(A^*-A) = X \cap (A^* \cap A^c)^c = X \cap ((A^*)^c \cup A) = (X \cap (A^*)^c) \cup (X \cap A) = A \cup (X-A^*) \), the equivalence is clear.

**Theorem 2.15.** Let \((X, \tau, \mathcal{I})\) be an ideal space. Then every subset of \(X\) is \(\mathcal{I}_{sg}\)-closed if and only if every \(\hat{g}\)-open set is \(\ast\)-closed.

**Proof.** Suppose every subset of \(X\) is \(\mathcal{I}_{sg}\)-closed. If \(U \subseteq X\) is \(\hat{g}\)-open, then \(U\) is \(\mathcal{I}_{sg}\)-closed and so \(U^* \subseteq U\). Hence \(U\) is \(\ast\)-closed.

Conversely, suppose that every \(\hat{g}\)-open set is \(\ast\)-closed. If \(U\) is \(\hat{g}\)-open set such that \(A \subseteq U \subseteq X\), then \(A^* \subseteq U^* \subseteq U\) and so \(A\) is \(\mathcal{I}_{sg}\)-closed.

The following theorem gives a characterization of normal spaces in terms of \(\mathcal{I}_{sg}\)-open sets.

**Theorem 2.16.** Let \((X, \tau, \mathcal{I})\) be an ideal space where \(\mathcal{I}\) is completely codense. Then the following are equivalent.

1. \(X\) is normal,
2. For any disjoint closed sets \(A\) and \(B\), there exist disjoint \(\mathcal{I}_{sg}\)-open sets \(U\) and \(V\) such that \(A \subseteq U\) and \(B \subseteq V\),
3. For any closed set \(A\) and open set \(V\) containing \(A\), there exists an \(\mathcal{I}_{sg}\)-open set \(U\) such that \(A \subseteq U \subseteq cl^*(U) \subseteq V\).

**Proof.** (1)⇒(2) The proof follows from the fact that every open set is \(\mathcal{I}_{sg}\)-open.

(2)⇒(3) Suppose \(A\) is closed and \(V\) is an open set containing \(A\). Since \(A\) and \(X-V\) are disjoint closed sets, there exist disjoint \(\mathcal{I}_{sg}\)-open sets \(U\) and \(W\) such that \(A \subseteq U\) and \(X-V \subseteq W\). Since \(X-V\) is \(\hat{g}\)-closed and \(W\) is \(\mathcal{I}_{sg}\)-open, \(X-V \subseteq int^*(W)\) and so \(X-int^*(W) \subseteq V\). Again \(U \cap W = \emptyset\) which implies that \(U \cap int^*(W) = \emptyset\) and so \(U \subseteq X-int^*(W)\) which implies that \(cl^*(U) \subseteq X-int^*(W) \subseteq V\). \(U\) is the required \(\mathcal{I}_{sg}\)-open sets with \(A \subseteq U \subseteq cl^*(U) \subseteq V\).
(3) ⇒ (1) Let A and B be two disjoint closed subsets of X. By hypothesis, there exists an \( I_g \)-open set U such that \( A \subseteq U \subseteq \text{cl}^*(U) \subseteq X - B \). Since U is \( I_g \)-open, \( A \subseteq \text{int}^*(U) \).

Since \( I \) is completely codense, by Lemma 1.4, \( \tau^* \subseteq \tau^a \) and so \( \text{int}^*(U) \) and \( X - \text{cl}^*(U) \in \tau^a \). Hence \( A \subseteq \text{int}^*(U) \subseteq \text{int}(\text{cl}(\text{int}(\text{int}^*(U)))) = G \) and \( B \subseteq X - \text{cl}^*(U) \subseteq \text{int}(\text{cl}(\text{int}(X - \text{cl}^*(U)))) = H \). G and H are the required disjoint open sets containing A and B respectively, which proves (1).

**Definition 2.2.** A subset A of an ideal space \((X, \tau, I)\) is said to be an \( \alpha \hat{g} \)-closed set [1] if \( \text{cl}^\alpha(A) \subseteq U \) whenever \( A \subseteq U \) and U is \( \hat{g} \)-open. The complement of \( \alpha \hat{g} \)-closed is said to be an \( \alpha \hat{g} \)-open set.

If \( I = \mathcal{N} \), it is not difficult to see that \( I_g \)-closed sets coincide with \( \alpha \hat{g} \)-closed sets and so we have the following Corollary.

**Corollary 2.7.** Let \((X, \tau, I)\) be an ideal space where \( I = \mathcal{N} \). Then the following are equivalent.

1. \( X \) is normal,
2. For any disjoint closed sets A and B, there exist disjoint \( \alpha \hat{g} \)-open sets U and V such that \( A \subseteq U \) and \( B \subseteq V \),
3. For any closed set A and open set V containing A, there exists an \( \alpha \hat{g} \)-open set U such that \( A \subseteq U \subseteq \text{cl}_\alpha(U) \subseteq V \).

**Definition 2.3.** A subset A of an ideal space is said to be \( I \)-compact [4] or compact modulo \( I \) [13] if for every open cover \( \{U_\alpha \mid \alpha \in \Delta\} \) of A, there exists a finite subset \( \Delta_0 \) of \( \Delta \) such that \( A - \bigcup \{U_\alpha \mid \alpha \in \Delta_0\} \in \mathcal{I} \). The space \((X, \tau, I)\) is \( I \)-compact if X is \( I \)-compact as a subset.

**Theorem 2.17.** Let \((X, \tau, I)\) be an ideal space. If A is an \( I_g \)-closed subset of X, then A is \( I \)-compact [12, Theorem 2.17].
Corollary 2.8. Let \((X,\tau,\mathcal{I})\) be an ideal space. If \(A\) is an \(\mathcal{I}_g\)-closed subset of \(X\), then \(A\) is \(\mathcal{I}\)-compact.

Proof. The proof follows from the fact that every \(\mathcal{I}_g\)-closed is \(\mathcal{I}_g\)-closed.

Acknowledgement

We thank the referees for their suggestions for improvement of this paper.

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(1) Department of Mathematics, P. M. Thevar College, Usilampatti, Madurai, Tamil Nadu, India.
E-mail address: siingam@yahoo.com

(2) Department of Mathematics, R. V. S College of Engineering and Technology, Dindigul, Tamil Nadu, India.
E-mail address: tharmar11@yahoo.co.in

(3) Department of Mathematics, Yadava College, Madurai, Tamil Nadu, India.
E-mail address: sangeethaabi10@gmail.com

(4) Department of Mathematics, V. O. Chidambaram College, Thoothukudi, Tamil Nadu, India.
E-mail address: antonyrexrodrigo@yahoo.co.in