A DECOMPOSITION OF $(\mu, \lambda)$-CONTINUITY IN GENERALIZED TOPOLOGICAL SPACES

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Abstract. In this paper, we introduce and study the notions of $w_{(\mu, \lambda)} - \mathcal{H}$-continuity and $w^*_{(\mu, \lambda)} - \mathcal{H}$-continuity in generalized topological spaces. Also, we prove that $f : (X, \mu) \rightarrow (Y, \lambda, \mathcal{H})$ is $(\mu, \lambda)$-continuous if and only if it is $w_{(\mu, \lambda)} - \mathcal{H}$-continuous and $w^*_{(\mu, \lambda)} - \mathcal{H}$-continuous.

1. Introduction and Preliminaries

In 2002, Csaszar\cite{2} introduced the notions of generalized topology and generalized continuity. Let $X$ be a nonempty set and $\mu$ be a collection of subsets of $X$. Then $\mu$ is called a generalized topology (briefly GT) on $X$ iff $\emptyset \in \mu$ and the union of an arbitrary class of elements of $\mu$ always belong to $\mu$. We call the pair $(X, \mu)$ be a generalized topological space (briefly GTS) on $X$. Let $\mu$ be a GT in $X$. The elements of $\mu$ are said to be $\mu$-open, their complements are $\mu$-closed. We consider the largest $\mu$-open subset of $A \subset X$ and denote it by $i_\mu(A)$ and the smallest $\mu$-closed superset of $A$ and denoted it by $c_\mu(A)$. A subset $A$ of $X$ is $\mu$-pre-open \cite{3} (resp. $\mu$-semi-open \cite{3}), if $A \subset i_\mu c_\mu(A)$ (resp. $A \subset c_\mu i_\mu(A)$). A

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generalized topological space \((X, \mu)\) is said to be \(\mu\)-regular [8] if for each \(\mu\)-closed set \(F\) of \(X\) not containing \(x\), there exist disjoint \(\mu\)-open sets \(U\) and \(V\) such that \(x \in U\) and \(F \subseteq V\). A function \(f : (X, \mu) \to (Y, \lambda)\) is said to be \((\mu, \lambda)\)-continuous [2] (resp. \((\pi, \lambda)\)-continuous [7]), iff \(U \in \lambda\) implies that \(f^{-1}(U)\) is \(\mu\)-open (resp. \(\mu\)-pre-open) in \((X, \mu)\). A function \(f : (X, \mu) \to (Y, \lambda)\) is said to be weakly \((\mu, \lambda)\)-continuous [6], if for each \(x \in X\) and each \(\lambda\)-open neighbourhood \(V\) of \(f(x)\), there exist a \(\mu\)-open neighbourhood \(U\) of \(x\) such that \(f(U) \subseteq c_\lambda(V)\).

A nonempty family \(\mathcal{H}\) of subsets of \(X\) is said to be a hereditary class [4], if \(A \in \mathcal{H}\) and \(B \subset A\), then \(B \in \mathcal{H}\). Given a generalized topological space \((X, \mu)\) with a hereditary class \(\mathcal{H}\), for each \(A \subseteq X\), \(A^*(\mathcal{H}, \mu) = \{x \in X : A \cap V \notin \mathcal{H} \text{ for every } V \in \mu \text{ such that } x \in V\}\) [4]. If \(c_\mu^*(A) = A \cup A^*(\mathcal{H}, \mu)\) for every subset \(A\) of \(X\), then \(\mu^* = \{A \subset X : X - A = c_\mu^*(X - A)\}\) is a GT, \(\mu^*\) is finer than \(\mu\) [[4], Theorem 3.6]. A subset \(A\) of \((X, \mu, \mathcal{H})\) is said to be pre-\(\mathcal{H}\)-open [6], if \(A \subset i_\mu c_\mu^*(A)\). A hereditary class \(\mathcal{H}\) is \(\mu\)-codense [4], iff \(\mu \cap \mathcal{H} = \{\emptyset\}\). A hereditary class \(\mathcal{H}\) is strongly \(\mu\)-codense [4], iff \(M, M' \in \mu, M \cap M' \in \mathcal{H}\) implies \(M \cap M' = \emptyset\).

**Definition 1.1.** [2] A function \(f : (X, \mu) \to (Y, \lambda)\) is said to be \(\theta(\mu, \lambda)\)-continuous at \(x\) if for each \(\lambda\)-open neighbourhood \(V\) of \(f(x)\), there is a \(\mu\)-open neighbourhood \(U\) of \(x\) such that \(f(c_\mu(U)) \subseteq c_\lambda(V)\).

**Lemma 1.2.** [[4], Proposition 2.8] Let \((X, \mu, \mathcal{H})\) be a hereditary generalized topological space. Then \(A \in \mu\) implies \(A \subseteq A^*\) iff \(\mathcal{H}\) is strongly \(\mu\)-codense.

**Lemma 1.3.** [4] Let \((X, \mu, \mathcal{H})\) be a hereditary generalized topological space and \(A, B\) be subsets of \(X\). Then the following properties are hold:

1. If \(A \subset B\), then \(A^* \subset B^*\),
2. \(A^* = c_\mu(A^*) \subseteq c_\mu(A)\),
3. \((A^*)^* \subseteq A^*\).
Lemma 1.4. [[4], Proposition 3.7] Let \((X, \mu, \mathcal{H})\) be a hereditary generalized topological space and \(A \subseteq X\). Then the following statements are equivalent.

1. \(A \subseteq A^*\),
2. \(A^* = c^*_\mu(A)\),
3. \(A^* = c_\mu(A)\).

Lemma 1.5. [[6], Theorem 3.2] Let \((X, \mu)\) and \((Y, \lambda)\) be generalized topological spaces. Then \(f : (X, \mu) \to (Y, \lambda)\) is \((\mu, \lambda)\)-continuous iff for each \(x \in X\) and each \(\lambda\)-open set \(V\) containing \(f(x)\), there exist a \(\mu\)-open set \(U\) containing \(x\) such that \(f(U) \subseteq V\).

Lemma 1.6. [[8], Theorem 4.3] Let \((X, \mu)\) be a generalized topological space. If \(X\) is \(\mu\)-regular, then for each \(x \in X\) and each \(U \in \mu\) containing \(x\), there exists \(V \in \mu\) such that \(x \in V \subseteq c_\mu(V) \subseteq U\).

2. Weakly \((\mu, \lambda)\)-continuity and weak* \((\mu, \lambda)\)-continuity

Definition 2.1. A function \(f : (X, \mu) \to (Y, \lambda, \mathcal{H})\) is said to be weakly \((\mu, \lambda)\)-\(\mathcal{H}\)-continuous (briefly \(w_{(\mu, \lambda)}\)-\(\mathcal{H}\)-c), if for each \(x \in X\) and each \(\lambda\)-open neighbourhood \(V\) of \(f(x)\), there exist a \(\mu\)-open neighbourhood \(U\) of \(x\) such that \(f(U) \subseteq c^*_\lambda(V)\).

Remark 2.2. Every weakly \((\mu, \lambda)\)-\(\mathcal{H}\)-continuous function is weakly \((\mu, \lambda)\)-continuous but the converse is need not be true.

Example 2.3. Let \(X = Y = \{a, b, c, d\}\), \(\mu = \emptyset, \{a, b\}, \{c, d\}, X\), \(\lambda = \emptyset, \{a, c, d\}, \{b, c, d\}, Y\), and \(\mathcal{H} = \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, c\}, \{c, d\}\). The identity function \(f : (X, \mu) \to (Y, \lambda, \mathcal{H})\) is weakly \((\mu, \lambda)\)-continuous but not weakly \((\mu, \lambda)\)-\(\mathcal{H}\)-continuous.
(i) Let $a \in X$. Then $V = \{a, c, d\}$ and $Y$ are the $\lambda$-open sets containing $f(a)$ in $(Y, \lambda)$. There exist a $\mu$-open set $U = \{a, b\}$ containing $a$ in $(X, \mu)$ such that $f(U) \subseteq c_\lambda(V) = Y$.

(ii) Let $b \in X$. Then $V = \{b, c, d\}$ and $Y$ are the $\lambda$-open sets containing $f(b)$ in $(Y, \lambda)$. There exist a $\mu$-open set $U = \{a, b\}$ containing $b$ in $(X, \mu)$ such that $f(U) \subseteq c_\lambda(V) = Y$.

(iii) Let $c \in X$. Then $V_1 = \{a, c, d\}$, $V_2 = \{b, c, d\}$ and $Y$ are the $\lambda$-open sets containing $f(c)$ in $(Y, \lambda)$. There exist a $\mu$-open set $U = \{c, d\}$ containing $c$ in $(X, \mu)$ such that $f(U) \subseteq c_\lambda(V) = Y$, where $V$ be a $\lambda$-open set containing $f(c)$.

(iv) Let $d \in X$. Then $V_1 = \{a, c, d\}$, $V_2 = \{b, c, d\}$ and $Y$ are the $\lambda$-open sets containing $f(d)$ in $(Y, \lambda)$. There exist a $\mu$-open set $U = \{c, d\}$ containing $d$ in $(X, \mu)$ such that $f(U) \subseteq c_\lambda(V) = Y$, where $V$ be a $\lambda$-open set containing $f(d)$.

By (i), (ii), (iii), and (iv), $f$ is weakly $(\mu, \lambda)$-continuous. On the other hand, consider the $\lambda$-open set $V = \{a, c, d\}$ in $(Y, \lambda)$. Now, $\{a, c, d\}^* = \{a\}$ and so $c_\lambda^*(\{a, c, d\}) = \{a, c, d\}$. Note that the $\mu$-open subsets of $(X, \mu)$ containing $a$ are $U = \{a, b\}$ and $X$. Further $f(U) \notin c_\lambda^*(V)$ and $f(X) \notin c_\lambda^*(V)$. Therefore $f$ is not $w_{(\mu, \lambda)}$-$\mathcal{H}$-c.

**Theorem 2.4.** A function $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$ is weakly $(\mu, \lambda)$-$\mathcal{H}$-continuous if and only if for each $\lambda$-open set $V \subseteq Y$, $f^{-1}(V) \subseteq i_\mu(f^{-1}(c_\lambda^*(V)))$.

**Proof.** Let $V$ be any $\lambda$-open set of $Y$ and $x \in f^{-1}(V)$. Since $f$ is $w_{(\mu, \lambda)}$-$\mathcal{H}$-c, there exists a $\mu$-open set $U$ such that $x \in U$ and $f(U) \subseteq c_\lambda^*(V)$. Hence $x \in U \subseteq f^{-1}(c_\lambda^*(V))$ and $x \in i_\mu(f^{-1}(c_\lambda^*(V)))$. Therefore, we obtain $f^{-1}(V) \subseteq i_\mu(f^{-1}(c_\lambda^*(V)))$.

Conversely, let $x \in X$ and $V$ be a $\lambda$-open set of $Y$ containing $f(x)$. Then $x \in f^{-1}(V) \subseteq i_\mu(f^{-1}(c_\lambda^*(V)))$. Let $U = i_\mu(f^{-1}(c_\lambda^*(V)))$,
then \( f(U) = f(i_\mu(f^{-1}(c_\lambda^*(V)))) \subset f(f^{-1}(c_\lambda^*(V))) \subset c_\lambda^*(V) \). This shows that \( f \) is \( w_{(\mu,\lambda)} - \mathcal{H} \)-c.

**Theorem 2.5.** Let \((Y,\lambda,\mathcal{H})\) be a hereditary generalized topological space, where \(\mathcal{H}\) is strongly \(\lambda\)-codense. Then the following are equivalent:

(a) \( f : (X,\mu) \to (Y,\lambda,\mathcal{H}) \) is weakly \((\mu,\lambda) - \mathcal{H}\)-continuous,

(b) For every \(\lambda\)-semi-open set \( V \) in \( Y \), there exist a \( \lambda \)-open set \( W \) in \( Y \) such that \( W \subset V \) and \( f^{-1}(W) \subset i_\mu(f^{-1}(V^*)) \),

(c) \( f^{-1}(W) \subset i_\mu(f^{-1}(V^*)) \) for every \( \lambda \)-open set \( V \) in \( Y \).

**Proof.** (a) \( \Rightarrow \) (b). Assume that \( f \) is weakly \((\mu,\lambda) - \mathcal{H}\)-continuous and \( V \) is \( \lambda\)-semi-open in \((Y,\lambda)\). Since \( V \) is \( \lambda\)-semi-open in \((Y,\lambda)\), there exist a \( \lambda \)-open set \( W \) in \((Y,\lambda)\) such that \( W \subset V \subset c_\lambda(W) \). Since \(\mathcal{H}\) is strongly \(\mu\)-codense, \( W^* = c_\lambda(W) = c_\lambda^*(W) \) by Lemmas 1.2 and 1.3. Therefore, \( W \subset V \subset W^* \) so that \( W^* = V^* = c_\lambda^*(W) \). By Theorem 2.4, \( f^{-1}(W) \subset i_\mu(f^{-1}(c_\lambda^*(W))) = i_\mu(f^{-1}(V^*)) \), which proves (b).

(b) \( \Rightarrow \) (c). Let \( V \) be \( \lambda\)-semi-open in \((Y,\lambda)\), there exist a \( \lambda \)-open set \( W \) in \((Y,\lambda)\) such that \( W \subset V \) and \( f^{-1}(W) \subset i_\mu(f^{-1}(V^*)) \). The set \( W \) be \( \lambda \)-open in \((Y,\lambda)\), then \( f^{-1}(W) \subset i_\mu(f^{-1}(W^*)) \).

(c) \( \Rightarrow \) (a). Let \( V \) be \( \lambda \)-open set in \((Y,\lambda)\). Then \( f^{-1}(V) \subset i_\mu(f^{-1}(V^*)) \). Since \(\mathcal{H}\) is strongly \(\mu\)-codense, then \( f^{-1}(V) \subset i_\mu(f^{-1}(c_\lambda^*(V))) \). By Theorem 2.4 \( f : (X,\mu) \to (Y,\lambda,\mathcal{H}) \) is weakly \((\mu,\lambda) - \mathcal{H}\)-continuous.

**Theorem 2.6.** If \((Y,\lambda,\mathcal{H})\) is a hereditary generalized topological space such that \(\mathcal{H}\) is strongly \(\lambda\)-codense and \( f : (X,\mu) \to (Y,\lambda,\mathcal{H}) \) is \( w_{(\mu,\lambda)} - \mathcal{H} \)-c, then \( c_\mu(f^{-1}(V)) \subset f^{-1}(c_\lambda^*(V)) = f^{-1}(V^*) \) for every \( \lambda \)-open set \( V \) in \( Y \).
Proof. Let $x \in c_\mu(f^{-1}(V))$. Assume that $x \notin f^{-1}(c_\lambda^*(V))$. By Lemmas 1.2 and 1.3, we have $x \notin f^{-1}(V^*)$ and $f(x) \notin V = c_\lambda(V)$. Therefore, there exist a $\lambda$-open set $W$ containing $f(x)$ such that $W \cap V = \emptyset$ which implies that $c_\lambda(W) \cap V = \emptyset$ and so $c_\lambda^*(W) \cap V = \emptyset$. Since $f$ is weakly $(\mu, \lambda)$-H-c, there is a $\mu$-open set $U$ containing $x$ in $X$ such that $f(U) \subset c_\lambda^*(W)$ and so $f(U) \cap V = \emptyset$. Now $x \in c_\mu(f^{-1}(V))$ implies that $U \cap f^{-1}(V) \neq \emptyset$ and so $f(U) \cap V \neq \emptyset$, a contradiction, which completes the proof.

If $\mathcal{H} = \emptyset$, in the above Theorem 2.6, we have the following Corollary.

Corollary 2.7. ([6], Theorem 3.5). If $(X, \mu)$ and $(Y, \lambda)$ are generalized topological spaces and $f : (X, \mu) \to (Y, \lambda)$ is weakly $(\mu, \lambda)$-continuous, then $c_\mu(f^{-1}(V)) \subset f^{-1}(c_\lambda(V))$, for every $\lambda$-open set $V$ in $(Y, \lambda)$.

Definition 2.8. A hereditary generalized topological space $(X, \mu, \mathcal{H})$ is $R_\mu\mathcal{H}$-space if, for each $x \in X$ and each $\mu$-open neighbourhood $U$ of $x$, there exist a $\mu$-open neighbourhood $V$ of $x$ such that $x \in V \subset c_\mu^*(V) \subset U$.

Theorem 2.9. Let $(Y, \lambda, \mathcal{H})$ be a $R_\mu\mathcal{H}$-space. Then $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$ is $w(\mu, \lambda)$-H-c if and only if $f$ is $(\mu, \lambda)$-continuous.

Proof. Let $x \in X$ and $V$ be a $\lambda$-open set of $Y$ containing $f(x)$. Since $Y$ is a $R_\mu\mathcal{H}$-space, there exist a $\lambda$-open set $W$ of $Y$ such that $f(x) \in W \subset c_\lambda^*(W) \subset V$. Let $f$ be $w(\mu, \lambda)$-H-c, there exist a $\mu$-open set $U$ such that $x \in U$ and $f(U) \subset c_\lambda^*(W) \subset V$. This implies $f$ is $(\mu, \lambda)$-continuous. Conversely, let $x \in X$ and $V$ be any $\lambda$-open set of $Y$ containing $f(x)$. Since $f$ is $(\mu, \lambda)$-continuous, there exist a $\mu$-open set $U$ containing $x$ such that $f(U) \subset V \subset c_\lambda^*(V)$, hence $f$ is $w(\mu, \lambda)$-H-c.
Theorem 2.10. If \((Y, \lambda, \mathcal{H})\) is a hereditary generalized topological space and \(f : (X, \mu) \rightarrow (Y, \lambda, \mathcal{H})\) is a \((\pi, \lambda)\)-continuous mapping such that 
\(c_\mu(f^{-1}(V)) \subset f^{-1}(c_\lambda^*(V))\) for every \(\lambda\)-open set \(V\) in \(Y\), then \(f\) is \(w_{(\mu, \lambda)}-\mathcal{H}\)-c.

Proof. Let \(x \in X\) and \(V\) be a \(\lambda\)-open set in \(Y\) containing \(f(x)\). By hypothesis 
\(c_\mu(f^{-1}(V)) \subset f^{-1}(c_\lambda^*(V))\). Since \(f\) is \((\pi, \lambda)\)-continuous, \(f^{-1}(V)\) is \(\mu\)-pre-open in \(X\) and so 
\(x \in f^{-1}(V) \subset i_\mu(c_\mu(f^{-1}(V)))\). Which implies there exist a \(\mu\)-open set such that 
\(x \in U \subset c_\mu(f^{-1}(V)) \subset f^{-1}(c_\lambda^*(V))\), that is 
\(U \subset f^{-1}(c_\lambda^*(V))\), \(f(U) \subset c_\lambda^*(V)\) which implies that \(f\) is \(w_{(\mu, \lambda)}-\mathcal{H}\)-c.

Theorem 2.11. If \((Y, \lambda, \mathcal{H})\) is a hereditary generalized topological space such that \(\mathcal{H}\) is strongly \(\lambda\)-codense and \(f : (X, \mu) \rightarrow (Y, \lambda, \mathcal{H})\) is \(\mu\)-pre-continuous, then \(f\) is \(w_{(\mu, \lambda)}-\mathcal{H}\)-c if and only if 
\(c_\mu(f^{-1}(V)) \subset f^{-1}(c_\lambda^*(V)) = f^{-1}(V^*)\) for every \(\lambda\)-open set \(V\) in \(Y\).

Proof. Follows from Theorem 2.6 and 2.10.

If \(f : (X, \mu, \mathcal{H}) \rightarrow (Y, \lambda)\) is any mapping. Then \(f(\mathcal{H}) = \{f(H) : H \in \mathcal{H}\}\) is a 
hereditary class on \((Y, \lambda)\).

Theorem 2.12. If \(f : (X, \mu, \mathcal{H}) \rightarrow (Y, \lambda, \mathcal{J})\) is a weakly \((\mu, \lambda)\)-\(\mathcal{J}\)-continuous 
where \(\mathcal{J} = f(\mathcal{H})\) is strongly \(\lambda\)-codense, then 
\(c_\mu^*(f^{-1}(V)) \subset f^{-1}(c_\lambda^*(V)) = f^{-1}(V^*)\) for every \(\lambda\)-open set \(V\) in \(Y\).

Proof. Let \(x \in c_\mu^*(f^{-1}(V))\). Assume that \(x \notin f^{-1}(c_\lambda^*(V))\). That is \(x \notin f^{-1}(V^*(\mathcal{J}))\) 
implies that \(f(x) \notin V^*(\mathcal{J}) = c_\lambda(V)\), since \(\mathcal{J}\) is strongly \(\lambda\)-codense. Therefore, 
there exist a \(\lambda\)-open set \(W\) containing \(f(x)\) in \(Y\) such that \(W \cap V = \emptyset\). Since \(V\) is 
\(\lambda\)-open, \(c_\lambda(W) \cap V = \emptyset\) and so \(c_\lambda^*(W) \cap V = \emptyset\). Since \(f\) is weakly \((\mu, \lambda)\)-\(\mathcal{J}\)-continuous, there exist a \(\mu\)-open set \(U\) in \(X\) containing \(x\) such that 
\(f(U) \subset c_\lambda^*(W)\)
and so \( f(U) \cap V = \emptyset \). Now, \( x \in c^*_\mu(f^{-1}(V)) \) implies that \( x \in c_\mu(f^{-1}(V)) \) which implies that \( f^{-1}(V) \cap U \neq \emptyset \) and so \( V \cap f(U) \neq \emptyset \), a contradiction. This completes the proof.

**Definition 2.13.** A function \( f : (X, \mu, \mathcal{H}) \to (Y, \lambda) \) is said to be pre-\( \mathcal{H} \)-continuous, if for each \( \lambda \)-open \( V \) in \( Y \), \( f^{-1}(V) \) is pre-\( \mathcal{H} \)-open in \( (X, \mu, \mathcal{H}) \).

**Theorem 2.14.** If \( f : (X, \mu, \mathcal{H}) \to (Y, \lambda, \mathcal{J}) \) is a pre-\( \mathcal{H} \)-continuous mapping where \( \mathcal{J} = f(\mathcal{H}) \) and \( c^*_\mu(f^{-1}(V)) \subset f^{-1}(c^*_\lambda(V)) \) for every \( \lambda \)-open set \( V \) in \( Y \), then \( f \) is weakly \( (\mu, \lambda) \)-\( \mathcal{J} \)-continuous.

**Proof.** Let \( x \in X \) and \( V \) be a \( \lambda \)-open set in \( Y \) containing \( f(x) \). By hypothesis, \( c^*_\mu(f^{-1}(V)) \subset f^{-1}(c^*_\lambda(V)) \). Since \( f \) is pre-\( \mathcal{H} \)-continuous, \( f^{-1}(V) \) is pre-\( \mathcal{H} \)-open in \( X \) and so \( f^{-1}(V) \subset i_\mu(c^*_\mu(f^{-1}(V))) \). Since \( x \in f^{-1}(V) \subset i_\mu(c^*_\mu(f^{-1}(V))) \), there exist a \( \mu \)-open set \( U \) containing \( x \) such that \( x \in U \subset c^*_\mu(f^{-1}(V)) \subset f^{-1}(c^*_\lambda(V)) \) and so \( f(U) \subset c^*_\lambda(V) \) which implies that \( f \) is weakly \( (\mu, \lambda) \)-\( \mathcal{J} \)-continuous.

**Definition 2.15.** A function \( f : (X, \mu) \to (Y, \lambda, \mathcal{H}) \) is said to be weak* \( (\mu, \lambda) \)-\( \mathcal{H} \)-continuous (briefly \( w^*_\mu(\lambda, \mathcal{H}) \)-\( \mathcal{H} \)-c), if for each \( \lambda \)-open set \( V \) in \( Y \), \( f^{-1}(f^*_\lambda(V)) \) is \( \mu \)-closed in \( (X, \mu) \), where \( f^*_\lambda(V) = V^* - i_\lambda(V) \) is \( \lambda \)-closed in \( (Y, \lambda, \mathcal{H}) \).

**Theorem 2.16.** A function \( f : (X, \mu) \to (Y, \lambda, \mathcal{H}) \) is \( (\mu, \lambda) \)-continuous if and only if it is both \( w_\mu(\lambda, \mathcal{H}) \)-\( \mathcal{H} \)-c and \( w^*_\mu(\lambda, \mathcal{H}) \)-\( \mathcal{H} \)-c.

**Proof.** Let \( x \in X \) and \( V \) be any \( \lambda \)-open set of \( Y \) containing \( f(x) \). Since \( f \) is \( (\mu, \lambda) \)-continuous, there exist a \( \mu \)-open set \( U \) containing \( x \) such that \( f(U) \subset V \subset c^*_\lambda(V) \) and \( f^{-1}(f^*_\lambda(V)) \) is \( \mu \)-closed in \( (X, \mu) \). Hence \( f \) is \( w_\mu(\lambda, \mathcal{H}) \)-\( \mathcal{H} \)-c and \( w^*_\mu(\lambda, \mathcal{H}) \)-\( \mathcal{H} \)-c. Conversely, let \( x \in X \) and \( V \) be any \( \lambda \)-open set of \( Y \) containing \( f(x) \), since \( f \) is \( w_\mu(\lambda, \mathcal{H}) \)-\( \mathcal{H} \)-c, there exist a \( \mu \)-open set \( U \) containing
Example 2.18. Let $X = Y = \{a, b, c, d\}$, $\mu = \emptyset, \{a, b\}, \{c\}, \{a, b, c\}, \{b, c, d\}$, $X$, $\lambda = \emptyset, \{a, b\}, \{a, c, d\}, Y$, and $H = \emptyset, \{d\}$. The identity function $f : (X, \mu) \rightarrow (Y, \lambda, H)$ is $w_{(\mu, \lambda)} - H - c$ but not $w^*_{(\mu, \lambda)} - H - c$.

(i) Let $a \in X$. Then $V_1 = \{a, b\}$, $V_2 = \{a, c, d\}$ and $Y$ are the $\lambda$-open sets containing $f(a)$ in $(Y, \lambda)$. Now $V_1^* = V_2^* = Y$ and $c_\lambda(V_1) = c_\lambda(V_2) = Y$. There exist a $\mu$-open set $U = \{a, b\}$ of $(X, \mu)$ containing $a$ such that $f(U) \subset c_\lambda(V)$, where $V$ is a $\lambda$-open set in $(Y, \lambda)$ containing $f(a)$.

(ii) Let $b \in X$. Then $V = \{a, b\}$ and $Y$ are the $\lambda$-open sets containing $f(b)$ in $(Y, \lambda)$. Now $V^* = Y$ and $c_\lambda(V) = Y$. There exist a $\mu$-open set $U = \{a, b\}$ of $(X, \mu)$ containing $b$ such that $f(U) \subset c_\lambda(V)$.

(iii) Let $c \in X$. Then $V = \{a, c, d\}$ and $Y$ are the $\lambda$-open sets containing $f(c)$ in $(Y, \lambda)$. Now $V^* = Y$ and $c_\lambda(V) = Y$. There exist a $\mu$-open set $U = \{a, b, c\}$ of $(X, \mu)$ containing $c$ such that $f(U) \subset c_\lambda(V)$.

(iv) Let $d \in X$. Then $V = \{a, c, d\}$ and $Y$ are the $\lambda$-open sets containing $f(d)$ in $(Y, \lambda)$. Now $V^* = Y$ and $c_\lambda(V) = Y$. There exist a $\mu$-open set $U = \{b, c, d\}$ of $(X, \mu)$ containing $d$ such that $f(U) \subset c_\lambda(V)$.
By $(i)$, $(ii)$, $(iii)$, and $(iv)$, $f$ is $w_{(\mu, \lambda)} - \mathcal{H}$-c. On the other hand, consider the $\lambda$-open set $V = \{a, c, d\}$ in $(Y, \lambda)$. Now, $f^*_r(V) = V^* - i_\lambda(V) = \{b\}$. Since $f^{-1}(f^*_r(V)) = \{b\}$ and $\{b\}$ is not $\mu$-closed in $(X, \mu)$, $f$ is not $w_{(\mu, \lambda)} - \mathcal{H}$-c.

**Example 2.19.** Let $X = Y = \{a, b, c\}$, $\mu = \{\emptyset, \{a, b\}, \{c\}, X\}$, $\lambda = \{\emptyset, \{a\}, \{b, c\}, Y\}$ and $\mathcal{H} = \{\emptyset, \{c\}\}$. The identity function $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$ is $w_{(\mu, \lambda)} - \mathcal{H}$-c but not $w_{(\mu, \lambda)} - \mathcal{H}$-c.

(i) Let $a \in X$. Then $V = \{a\}$ and $Y$ are the $\lambda$-open sets containing $f(a)$ in $(Y, \lambda)$. Now, $\{a\}^* = \{a\}$ and $f^*_r(V) = V^* - i_\lambda(V) = \emptyset$. Hence $f^{-1}(f^*_r(V)) = f^{-1}(\emptyset) = \emptyset$ and $\emptyset$ is $\mu$-closed in $(X, \mu)$.

(ii) Let $b \in X$. Then $V = \{b, c\}$ and $Y$ are the $\lambda$-open sets containing $f(b)$ in $(Y, \lambda)$. Now, $(\{b, c\})^* = \{b, c\}$ and $f^*_r(V) = V^* - i_\lambda(V) = \emptyset$. Hence $f^{-1}(f^*_r(V)) = f^{-1}(\emptyset) = \emptyset$ and $\emptyset$ is $\mu$-closed in $(X, \mu)$.

(iii) Let $c \in X$. Then $V = \{b, c\}$ and $Y$ are the $\lambda$-open sets containing $f(c)$ in $(Y, \lambda)$. Now, $(\{b, c\})^* = \{b, c\}$ and $f^*_r(V) = V^* - i_\lambda(V) = \emptyset$. Hence $f^{-1}(f^*_r(V)) = f^{-1}(\emptyset) = \emptyset$ and $\emptyset$ is $\mu$-closed in $(X, \mu)$.

By $(i)$, $(ii)$, and $(iii)$, $f$ is $w_{(\mu, \lambda)}^* - \mathcal{H}$-c. On the other hand, consider the $\lambda$-open sets $V = \{a\}$ and $(Y, \lambda)$ are containing $f(a)$ in $Y$. Now, $\{a\}^* = \{a\}$ and so, $c_\lambda^*(V) = \{a\}$. Note that the $\mu$-open sets of $(X, \mu)$ containing $a$ are $U = \{a, b\}$ and $X$. Further $f(U) = U \not\subseteq c_\lambda^*(V)$ and $f(X) = Y \not\subseteq c_\lambda^*(V)$. Therefore $f$ is not $w_{(\mu, \lambda)} - \mathcal{H}$-c.

**Definition 2.20.** A hereditary generalized topological space $(X, \mu, \mathcal{H})$ is said to be $F_{\mu, \mathcal{H}^*}$-space, if $c_{\mu}(U) \subseteq U^*$ for every $\mu$-open set $U \subseteq X$.

**Theorem 2.21.** Let $(X, \mu, \mathcal{H})$ be an $F_{\mu, \mathcal{H}^*}$-space and $A \in \mu$. Then the following properties are hold:
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(1) $A^* = c^*_\mu(A) = (c_\mu(A))^* = c_\mu(A^*) = c^*_\mu(A^*)$

(2) $c^*_\mu(c_\mu(A)) = c_\mu(c^*_\mu(A)) = c^*_\mu(A^*)$

Proof. 1. Let $(X, \mu, \mathcal{H})$ be an $F_\mu \mathcal{H}^*$ space and $A \in \mu$. Then $c_\mu(A) \subseteq A^*$. Thus $(c_\mu(A))^* \subseteq (A^*)^* \subseteq A^*$ by Lemma 1.3. Also $A \subseteq c_\mu(A)$, $A^* \subseteq (c_\mu(A))^*$ by Lemma 1.3. Therefore $A^* = (c_\mu(A))^*$. By Lemma 1.3, $A^* = c_\mu(A^*)$. Since $(X, \mu, \mathcal{H})$ is an $F_\mu \mathcal{H}^*$ space, $A^* \subseteq c^*_\mu(A) \subseteq c_\mu(A) \subseteq A^*$. Thus, $A^* = c^*_\mu(A) = c_\mu(A)$. Now, $c^*_\mu(A^*) = c^*_\mu(c_\mu(A)) = c_\mu(A) \cup (c_\mu(A))^* = A^* \cup A^* = A^*$. Hence we obtain $A^* = c^*_\mu(A) = (c_\mu(A))^* = c_\mu(A^*) = c^*_\mu(A^*)$.

(2) Follows from (1).

Lemma 2.22. If a hereditary generalized topological space $(Y, \lambda, \mathcal{H})$ is $F_\lambda \mathcal{H}^*$ space and a function $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$ is $w(\mu, \lambda)$-c then $c^*_\mu(f^{-1}(V)) \subseteq f^{-1}(c^*_\lambda(V))$ for each $\lambda$-open set $V \subseteq Y$.

Proof. Let $x \in c^*_\mu(f^{-1}(V))$. Assume that $x \notin f^{-1}(c^*_\lambda(V))$. Then $f(x) \notin c^*_\lambda(V)$, we have $f(x) \notin V$ and $f(x) \notin V^*$. Since $Y$ is $F_\lambda \mathcal{H}^*$ space, $f(x) \notin c_\lambda(V)$. Hence there exist a $\lambda$-open set $W$ containing $f(x)$ such that $W \cap V = \emptyset$. Since $V$ is $\lambda$-open, $V \cap c_\lambda(W) = \emptyset$ and hence we have $V \cap c^*_\lambda(W) = \emptyset$. Since $f$ is $w(\mu, \lambda)$-c, there exist a $\mu$-open set $U \subseteq X$ containing $x$ such that $f(U) \subseteq c^*_\lambda(W)$. Thus we obtain $f(U) \cap V = \emptyset$. On the other hand, $x \in c^*_\mu(f^{-1}(V))$ and we have $x \in c_\mu(f^{-1}(V))$ and hence $U \cap f^{-1}(V) \neq \emptyset$. Thus $f(W) \cap V \neq \emptyset$, a contradiction so, $c^*_\mu(f^{-1}(V)) \subseteq f^{-1}(c^*_\lambda(V))$.

Theorem 2.23. Let $(X, \mu)$ be a $\mu$-regular space and $(Y, \lambda, \mathcal{H})$ be an $F_\lambda \mathcal{H}^*$ space. A function $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$ is $\theta(\mu, \lambda)$-continuous if and only if it is $w(\mu, \lambda)$-c.
Proof. Let $f$ be $\theta(\mu, \lambda)$-continuous, $x \in X$ and $V$ be any $\lambda$-open set of $Y$ containing $f(x)$. Since $f$ is $\theta(\mu, \lambda)$-continuous, there exists a $\mu$-open neighbourhood $U$ of $x$ such that $f(c_\mu(U)) \subseteq c_\lambda(V)$. Since $(Y, \lambda, \mathcal{H})$ is an $F_\lambda\mathcal{H}^*$-space, $f(U) \subseteq f(c_\mu(U)) \subseteq c_\lambda(V) \subseteq V^* \subseteq V \cup V^* \subseteq c^*_\lambda(V)$. Thus $f$ is $w(\mu, \lambda)$-$\mathcal{H}$-c. Conversely, let $f$ be $w(\mu, \lambda)$-$\mathcal{H}$-c, $x \in X$ and $V$ be any $\lambda$-open set of $Y$ containing $f(x)$. Since $f$ is $w(\mu, \lambda)$-$\mathcal{H}$-c, there exists a $\mu$-open neighbourhood $U$ of $x$ such that $f(U) \subseteq c^*_\lambda(V)$. Since $\lambda \subseteq \lambda^*$, $f(U) \subseteq c^*_\lambda(V) \subseteq c_\lambda(V)$. Since $(X, \mu)$ is a $\mu$-regular space, there exists a $\mu$-open neighbourhood $W$ of $x$ such that $x \in W \subseteq c_\mu(W) \subseteq U$ by Lemma 1.6. Then $f(c_\mu(W)) \subseteq f(U) \subseteq c_\lambda(V)$. Thus $f$ is $\theta(\mu, \lambda)$-continuous.

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References


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