ON ALMOST WN-INJECTIVE RINGS

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ABSTRACT: Let $R$ be a ring. Let $M_R$ be a module with $S = \text{End}(M_R)$. The module $M$ is called almost W-nil-injective (briefly right AWN-injective) if, for any $0 \neq a \in N(R)$, there exists $n \geq 1$ and an $S$-submodule $X_a$ of $M$ such that $a^n \neq 0$ and $l_M(r_R(a^n)) = M^{a^n} \oplus X_a$, as left $S$-modules. If $R$ is almost W-nil-injective, then we call $R$ is right almost W-nil-injective ring. In this paper, we give some characterization and properties of almost W-nil-injective rings. In particular, Conditions under which right almost W-nil-injective rings are $n$-regular rings and $n$-weakly regular rings are given. Also we study rings whose simple singular right $R$-module are almost W-nil-injective. It is proved that if $R$ is a NCI ring, whose every simple singular $R$-module is almost W-nil-injective, Then $R$ is reduced.

1. INTRODUCTION

Throughout the paper $R$ is an associative ring with identity, and is a right $R$-module with $S = \text{End}(M_R)$. For $a \in R$, $r(a), l(a)$ denote the right annihilator and left annihilator of $a$, respectively. We write $J(R), Z(R), Y(R)$, for the Jacobson radical and the left (right) singular ideal of $R$, respectively. $X \leq M$ denote that $X$ is a submodule of $M$.

Following [9] a ring $R$ is called a right (left) NPP if for $aR$ is projective for all $a \in N(R)$ (the set of nilpotent elements). Clearly, right (left) PP ring (that is if every principal right ideal of $R$ is projective as right $R$-module) is right (left)NPP, but the converse is not true by [9]. The ring $R$ is said to be reduced if $R$ has no non zero nilpotent
element. The ring $R$ is called right (left) SXM [10], if for each $0 \neq a \in R$, $r(a) = r(a^n)$ [ $l(a) = l(a^n)$ ] for all positive integer $n$ satisfying $a^n \neq 0$. For example, reduced rings are right (left) SXM ring. $R$ is said to be Von Neumann regular (or just regular), $a \in aRa$ for every $a \in R$ [15], a ring $R$ is called $n$-regular [9] if $a \in aRa$ for all $a \in N(R)$. Clearly, Von Neumann regular ring are $n$-regular, but the converse is not true. A ring $R$ is called right (left) $n$-weakly regular if $a \in aRaR$ ( $a \in RaRa$ ), for all $a \in N(R)$ [4]. Call a ring $R$ right MC2 if for right minimal element $k \in R$, $kR$ is a summand in $R_k$, whenever $kR$ is projective as right $R$-module[8]. A ring $R$ is called weakly reversible if $ab = 0$ implies that $Rbra$ is a nil left ideal of $R$ for all $a, b, r \in R$ [14]. Generalizations of injectivity have been discussed in many papers see [5] ,[6] . A right $R$-module $M$ is called principal injective (or P-injective), if every $R$-homomorphism from a principal right ideal of $R$ to $M$ can be extended to an $R$-homomorphism from $R$ to $M$. Equivalently, $l_M r_k(a) = Ma$ for all $a \in R$ [2]. In [5], Nicholson and Yousif studied the structure of principally injective rings and give some applications. They also continued to study rings with some other kind of injectivity, namely, GP-injective rings [6] and [10]. A ring $R$ is called GP-injective if for any $a \in R$ there exists a positive integer $n$ with $a^n \neq 0$ and $lr(a^n) = Ra^n$. Right GP-injective rings are called right YJ-injective rings by several authors. In [18], Zhao introduced an almost P-injective module. Let $M_R$ be a right $R$-module with $S = \text{End}(M_R)$. The module $M$ is called AP-injective, if for any $a \in R$, there exists a left $S$-submodule $X_a$ of $M_R$ such that $l_M r_k(a) = Ma \oplus X_a$.

AP-injectivity has been generally studied (see [6]). In [9], Wei and Jianhua first introduced and characterized a right nil-injective ring, and give many properties. A ring $R$ is said to be reversible if $ab = 0$ implies that $ba = 0$ for all $a, b \in R$. A ring $R$ is called right nil-injective, if $a \in N(R)$, $lr(a) = Ra$. In [19], Zhao and Du introduced an almost nil-injective module. Let $M_R$ be a module with $S = \text{End}(M_R)$. The module $M$ is called right
almost nil-injective if for any \( k \in N(R) \), there exists an \( S \)-submodule \( X_k \) of \( M \) such that \( l_x r'_k (k) = M_k \otimes X_k \) as left \( S \)-module. If \( R \) is almost nil-injective then we call \( R \) a right almost nil-injective ring.

2. Characterizations of Almost \( W_n \)-Injective

In this section we introduced the notion of a right GNNP and almost \( W_n \)-injective with some of their basic properties; we also give necessary and sufficient conditions for almost \( W_n \)-injective to be \( n \)-regular.

Following [9] a right \( R \)-module \( M \) is called \( W_n \)-injective, if for any \( 0 \neq a \in N(R) \), there exists a positive integer \( n \) such that \( a^n \neq 0 \) and any right \( R \)-homomorphism \( f : a^n R \rightarrow M \) can be extends to \( R \rightarrow M \). Equivalently, if for any \( 0 \neq a \in N(R) \) there exists a positive integer \( n \) such that \( a^n \neq 0 \) and \( Ra^n = lr(a^n) \).

Clearly right nil-injective module are all \( W_n \)-injective module.

Remark [6]:

We fix the following notation. If \( N \) is a submodule of \( M \), we write \( N/M \) to indicate that \( N \) is a direct summand of \( M \). For an \( (R,R) \)-bimodule \( M \), we let \( R \alpha M \) be the trivial extension of \( R \) and \( M \), i.e., \( R \alpha M = R \oplus M \) as an abelian group, with the following multiplication:

\[
(r,x) (s,y) = (rs, ry + xs)
\]

Example 6:

A non commutative right almost nil-injective ring which is not a right \( W_n \)-injective.

Let \( C \) be a noncommutative division subring of a division ring \( D \) such that the \( C \)-vector space \( cD \) has dimension \( >1 \). Let \( R = C \alpha D \) be the trivial extension of \( C \) and the \( C \)-module \( D \). Then \( R \) is not commutative. Let \( 0 \neq a = (c,d) \in N(R) \). If \( c \neq 0 \) then \( a \) is invertible in \( R \) and so we can let \( X_a = (0) \). If \( c = 0 \), then \( lr(a) = (0) \alpha D \) and \( Ra = (0) \alpha Cd \). Write \( D = Cd \oplus D_1 \) as a left \( C \)-vector space and let \( X_a = (0) \alpha D_1 \). Then
\( lr(a) = Ra \oplus X_a \). Therefore, \( R \) is right almost nil-injective. Note that \( a^2 = 0 \) and \( lr(a) \neq Ra \). Thus \( R \) is not right \( W \) nil-injective.

**Lemma 2.1 [11]:**

The following conditions are equivalent for a ring \( R \):

1. \( R \) is \( n \)-regular.
2. Every right \( R \)-module is \( W \)nil-injective.
3. Every cyclic right \( R \)-module is \( W \)nil-injective.
4. \( R \) is right \( W \)nil-injective and NPP ring.

**Lemma 2.2 [18]:**

Suppose \( M \) is a right \( R \)-module with \( S = \text{End}(M_R) \). If \( l_M r_R(a) = Ma \oplus X_a \), where \( X_a \) is a left \( S \)-submodule of \( M_R \). Set \( f : aR \to M \) is a right \( R \)-homomorphism, then \( f(a) = ma + x \) with \( m \in M \), \( x \in X_a \).

Now we give the following definition.

**Definition 2.3:**

A ring \( R \) is said to be right (left) GNPP if \( (Ra)^n \) is projective for all \( a \in N(R) \) and for some positive integer \( n \), \( a^n \neq 0 \).

Clearly every \( n \)-regular rings, reduced rings and NPP are right GNPP rings.

**Lemma 2.4 [9]:**

If \( R \) is a right NPP ring, then \( Y(R) = 0 \).

As a parallel result to Lemma (2.4), the following result was obtained:

**Proposition 2.5:**

Let \( R \) be a right GNPP ring. Then \( Y(R) = 0 \).
Let $0 \neq a \in Y(R)$, with $a^2 = 0$. Then $a \in N(R)$. Since $R$ is a right GNPP ring, then there exists a positive integer $n$ such that $a^n \neq 0$, $a^n R$ is projective. But $a^2 = 0$, so $n = 1$ and $aR$ is projective. Thus $r(a)$ is a direct summand of $R$ as a right $R$-module. But $a \in Y(R)$, $r(a)$ must be essential in $R$, which is a contradiction. Hence $Y(R) = 0$.

According to [16], a ring $R$ is right GQ-injective if for any right ideal $I$ isomorphic to a complement right ideal of $R$, every right $R$-homomorphism of $I$ into $R$ extends to an endomorphism of $_RR$.

In [16], shows that if $R$ is right (left) GQ-injective, then $J(R) = Y(R)$ ($J = Z$), $R/J$ is regular.

Every regular ring is right (left) GQ-injective. Clearly, $R$ is regular if and only if $R$ is right (left) GQ-injective right non singular [12].

**Corollary 2.6:**

If $R$ is a right GNPP-ring, then $R$ is regular if and only if $R$ is right GQ-injective.

**Proof:**

Since $R$ is right GQ-injective then $Y(R) = J(R)$ and $R/J$ is regular ring. By Proposition (2.5) $0 = Y(R) = J(R)$, So $R$ is regular ring.

Conversely: It is clear.

Call a ring is right NC2 if $aR$ projective implies $aR = eR$, $e = e^2 \in R$ for all $a \in N(R)$ [11]. Every $n$-regular rings is NPP and NC2 rings [10].

**Proposition 2.7:**

If $R$ is a ring with $l(a^n) \subseteq l(a)$, then $R$ is right NC2 and GNPP if and only if $R$ is $n$-regular.

**Proof:**

Let $a \in N(R)$. Since $R$ is right GNPP, then $a^n R$ is projective for some positive integer $n$ and $a^n \neq 0$. Since $R$ is right NC2 ring, $a^n R = eR$. 
\[ e^2 = e \in R \] Thus \( a^n = ea^n \) implies that \( a = ea \ (l(a^n) \subseteq l(a)) \). So \( e = ab \) for some \( b \in R \). Hence \( a = ea = aba \in aRa \). Thus \( R \) is \( n \)-regular.

Conversely:

Let \( R \) is \( n \)-regular ring ,implies that \( R \) is NPP ring. So is GNPP and NC2 ring.

In [6], Stanley and Yiqiang introduced an almost generalized principally injective (AGP-injective) module. Let \( M \) be a right \( R \)-module with \( S = \text{End}(M_R) \). The module \( M \) is called AGP-injective if , for any \( 0 \neq a \in R \), there exists a positive integer \( n \) and S-submodule \( X_a \) of \( M \) such that \( a^n \neq 0 \) and \( l_M r_R(a^n) = M a^n \oplus X_a \) as a left \( S \)-modules. Also studied right AGP-injective rings and give some characterization and properties which generalization results of [19].

Now, we consider rings which are more general than WN-injective rings, an idea parallel to the notion of AGP-injective rings.

**Definition 2.8:**

Let \( M_R \) be a module with \( S = \text{End}(M_R) \). The module \( M \) is called almost WN-nil-injective (briefly right AWN-nil-injective) if , for any \( 0 \neq a \in N(R) \), there exists \( n \geq 1 \) and an S-submodule \( X_a \) of \( M \) such that \( a^n \neq 0 \) and \( l_M r_R(a^n) = M a^n \oplus X_a \) as left \( S \)-modules.

If \( R_R \) is almost \( WN \)-injective, then we call \( R \) is right almost \( WN \)-injective ring.

Remark:

\[
\{\text{right YJ-injective rings}\} \subset \{\text{right nil-injective rings}\} \subset \{\text{right WN-injective rings}\} \\
\subset \{\text{right AWN-injective rings}\} \\
\{\text{right AN-injective rings}\} \subset \{\text{right AWN-injective rings}\}
\]

**Examples [19]:**

The ring \( \mathbb{Z} \) of integers is AWN-injective which is not AGP-injective.
Let $Z_2$ be a field, and $R = \begin{bmatrix} Z_2 & Z_2 \\ 0 & Z_2 \end{bmatrix}$, $N(R) = \begin{bmatrix} 0 & Z_2 \\ 0 & 0 \end{bmatrix}$. Let $0 \neq u \in Z_2$, then

\[
\begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & Z_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\]

and so $R$ is not right WN-injective but $R$ is AWN-injective ($lr(a) = Ra \oplus X_a$).

Let $R = \begin{bmatrix} 0 & Z_2 \\ 0 & Z_2 \end{bmatrix}$, where $Z_2$ is a field. Then $N(R) = \begin{bmatrix} 0 & Z_2 \\ 0 & 0 \end{bmatrix}$. Let $y = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \in N(R)$, then $Ry = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $lr(y) = R$. Therefore $lr(y) \neq Ry$ and so $R$ is not WN-injective. But $lr(y) = Ry \oplus R$. So $R$ is right AWN-injective.

**Lemma 2.9 [3]:**

The following conditions are equivalent:

1- $R$ is n-regular.

2- $N_i(R) = \{ 0 \neq x \in R : x^2 = 0 \}$ is regular.

3- For any $a \in N(R)$, there exists a positive integer $n$ such that $a^n \neq 0$ and $a^nR$ is generated by idempotent.

It is clear that any n-regular rings is AWN-injective but the converse is not true.

The following Theorem gives a partial converse.

**Theorem 2.10:**

Let $R$ be a right SXM ring. Then the following conditions are equivalent:

1- $R$ is n-regular.

2- $R$ is a right AWN-injective right NPP-ring.

**Proof:**

(1) $\rightarrow$ (2) is clear by [Lemma 2.1]
(2) $\rightarrow$ (1). Let $0 \neq a \in N(R)$. Since $R$ is a right AWN-injective, then there exists $n \geq 1$ such that $a^n \neq 0$ and $lr(a^n) = Ra^n \oplus X_a$. Since $R$ is right NPP ring and $a^n \in N(R)$, $r(a^n) = (1-e)R$, $e^2 = e \in R$. Therefore $Re = lr(a^n) = Ra^n \oplus X_a$, $e = ra^n + x$, where $r \in R$, $x \in X_a$. So $a^n = a^n e = a^n ra^n + a^n x$, $(1-a^n r)a^n = a^n x \in Ra^n \cap X_a = 0$ and $a^n = a^n ra^n$ this implies that $(1-ba^n) \in r(a^n) = r(a)$ [ $R$ is SXM], yielding $a = a ra^n$. Take $c = ra^{n-1} \in R$, hence $a = ac a$. Therefore $R$ is n-regular.

**Proposition 2.11:**

Let $R$ be a ring whose every simple right $R$-module is AWN-injective then:

1. $J(R) \cap Soc(R) = 0$
2. $J(R)$ is a reduced ideal of $R$.

**Proof:**

If $J(R) \cap Soc(R) \neq 0$, then there exists a minimal right ideal $kR$ of $R$ with $kR \subseteq J(R)$. If $kR$ is a direct summand, then $kR = eR$ for some $0 \neq e^2 = e \in R$ and we get $e \in J(R)$, which is a contradiction. So that $(kR)^2 = 0$. Since $r(k)$ is maximal right ideal of $R$, then $R/r(k)$ is AWN-injective. Let $f : kR \rightarrow R/r(k)$ be defined by $f(kr) = r + r(k)$. Then $f$ is a well defined $R$-homomorphism. Since $R/r(k)$ is AWN-injective $l_{R/r(k)}(r(k)) = r(k)k \oplus X_k$ where $X_k$ is a left $S$-submodule of $M$. Therefore $1 + r(k) = f(k) = bk + r(k) + X$ (Lemma 2.2). Thus $1-bk + r(k) = x \in r(k)k \cap X_k = 0$, $1-bk \in r(k)$. Since $k \in J(R)$, then $bk \in J(R) \subseteq r(k)$, which implies $1 \in r(k)$, which is also a contradiction. Therefore $J(R) \cap Soc(R) = 0$. 


Let \( 0 \neq a \in J(R) \) such that \( a^2 = 0 \). Since \( a \neq 0 \), then there exists a maximal right ideal \( M \) of \( R \) containing \( r(a) \). Thus \( R/M \) is AWN-injective, and 
\[
l_{R/M} r(a) = (R/M)a \oplus X_a \leq R/M.
\]

Let \( f : aR \to R/M \) be defined by \( f(ar) = r + M \). Then \( f \) is a well defined \( R \)-homomorphism. So there exists \( r \in R, x \in X_a \) such that \( 1 + M = ra + M + x \), \( 1 - ra + M = x \in R/M \cap X_a = 0 \). Hence \( 1 - ra \in M \) and so \( 1 \in M \), which is a contradiction. Hence \( J(R) \) is reduced.

**Lemma 2.12 [1]:**

If \( Y(R) = 0 \), then \( SR \) is a maximal right quotient ring of \( R \). Thus the maximal right quotient ring of any right nonsingular ring is regular.

Now, we have the following theorem.

**Theorem 2.13:**

If \( R \) is a right GNPP right AWN-injective ring, then the center of \( R \) \((C(R))\) is \( n \)-regular.

**Proof:**

Since \( Y(R) = 0 \) [Proposition 2.5], then there exists a right maximal quotient ring \( S \) of \( R \) such that it is regular Lemma (2.12), then \( C(S) \) is regular [The center of a regular ring is regular]. For any \( 0 \neq a \in N(C(R)) \subseteq N(C(S)) \), there exists \( s \in C(S) \) such that \( a = asa = a^2s = sa^2 \). Thus \( r(a^n) = r(a), l(a) = l(a^n) \) for any positive integer \( n \). We Claim that \( a \) is \( n \)-regular in \( N(C(R)) \). Note that \( a^2 \neq 0 \), so there exists a positive integer \( m \) with \( a^{2m} \neq 0 \) such that \( lr(a^{2m}) = Ra^{2m} \oplus X_{a^{2m}} \) for some left ideal \( X_{a^{2m}} \) of \( R \) since \( R \) is right AWN-injective. Thus \( lr(a^{2m-1}) = lr(a^{2m}) = Ra^{2m} \oplus X_{a^{2m}} \) and So \( a^{2m-1} = da^{2m} + x \) for some \( d \in R \) and \( x \in X_{a^{2m}} \). Then \( a^{2m} = ada^{2m} + ax \) and \((1-ad)a^{2m} = ax \in Ra^{2m} \cap X_{a^{2m}} = 0 \).
Therefore \( (1-\text{ad})a^{2m} = 0 \) and \( (1-\text{ad}) \in l(a^{2m}) = l(a) \), and So \( a = \text{ada} = a^2 d \).

Let \( u = ad^2 \) then \( a = a^2 d = a(a^2 d)d = a^2 ad^2 = a^2 u \). For any \( x \in R \), \( a^2(xu-ux) = 0 \) So \( (xu-ux) \in r(a^2) = r(a) \), \( 0 = a(xu-ux) = a(xad^2-adx^2) = a^2(xd^2-d^2x) \), \( (xd^2-d^2x) \in r(a^2) = r(a) \).

Thus \( xu-ux = xad^2-adx^2 = a(xd^2-d^2x) = 0 \). So \( xu = ux \), \( u \in C(R) \) and \( a = auu \)

Therefore \( C(R) \) is n-regular.

**Lemma 2.14 [7]:**

If \( R \) is a semiprime ring , then \( r(a^n) = r(a) \) for any \( a \in C(R) \) and a positive integer \( n \).

**Proposition 2.15:**

If \( R \) is a semiprime right AWN-injective ring , then the center \( C(R) \) of \( R \) is n-regular.

**Proof:**

For any \( 0 \neq a \in N(C(R)) \), \( Ra \cap l(a) = 0 \). Since \( R \) is semiprime. Therefore, \( l(a^m) = l(c) = r(c) = r(a^n) \) for any a positive integer \( n \) Lemma (2.14). Note that \( a^2 = 0 \) because \( Ra \cap l(a) = 0 \). As in the proof of Theorem [2.13], \( C(R) \) is n-regular.

**Proposition 2.16:**

Let \( R \) be a ring , if for any element \( a \in N(R) \) , there exists a positive integer \( n \) such that \( r(a^n) \subseteq r(a) \) and \( a^n \neq 0 \) if \( R/r(a^n) \) is AWN-injective, then \( R \) is n-regular ring.

**Proof:**

Let \( a \) be any element in \( N(R) \) and let \( f : a^n R \to R/r(a^n) \) be defined by \( f(a^n s) = s + r(a^n) \) for all \( s \in R \) and positive integer \( n \) and \( a^n \neq 0 \). Then \( f \) is a well defined \( R \)-homomorphism. Since \( R/r(a^n) \) is AWN-injective, \( l_{R/r(a^n)} r_R(a^n) = (R/r(a^n)) a^n \oplus X_{a^n} \).
where \( X_{a^r} \) is a left \( S \)-submodule of \( R/r(a^n) \), \( (X_{a^r} \subseteq R) \). Then there exists \( b \in R \) and \( x \in X_{a^r} \) such that \( 1 + r(a^n) = f(a^n) = ba^n + r(a^n) + x \) (Lemma 2.2).

Thus \( 1 - ba^n + r(a^n) = x \in R/r(a^n) \cap X_{a^r} = 0 \), \( 1 - ba^n \in r(a^n) \subseteq r(a) \) implies that \( a = aba^n \). Take \( c = ba^{n-1} \), Hence \( a = aca \). Therefore \( R \) is n-regular ring.

**Theorem 2.17:**

Let \( R \) be a ring with \( a^nR = aR \) for every \( a \in R \) and a positive integer \( n \), \( a^n \neq 0 \). If every simple right \( R \)-module is AWN-injective, then \( R \) is right n-weakly regular ring.

**Proof:**

We will show that \( RaR + r(a) = R \) for any \( a \in N(R) \). If \( RaR + r(a) \neq R \), then there exists a maximal right ideal \( M \) of \( R \) containing \( RaR + r(a) \). Then \( R/M \) is AWN-injective, then \( l_{R/M}r(a^n) = (R/M)a^n \otimes X_{a^r} \), \( X_{a^r} \leq R/M \). Let \( f : a^nR \to R/M \) be defined by \( f(a^n r) = r + M \). Note that \( f \) is well defined. So \( 1 + M = f(a^n) = ca^n + M + x \), \( c \in R \), \( x \in X_{a^r} \), \( 1 - ca^n + M = x \in R/M \cap X = 0 \).

So \( 1 - ca^n \in M \). Since \( ca^n \in Ra^nR = RaR \subseteq M \), \( 1 \in M \), Which is a contradiction. Therefore \( RaR + r(a) = R \) for any \( a \in N(R) \), then \( R \) is a right n-weakly regular.

Following [13], a ring \( R \) is called right N duo if \( aR \) is an ideal of \( R \) for all \( a \in N(R) \).

Every reduced rings is N duo.

Now, we give the definition.

**Definition 2.18:**

An element \( x \in N(R) \) is called right (left) generalized n-regular if there exists a positive integer \( n \) such that \( x^n \neq 0 \) and \( x^n = x^n yx \) (\( x^n = xyx^n \)) for some \( y \in R \). A ring \( R \) is called right (left) generalized n-regular if every element in \( N(R) \) is right (left) generalized n-regular.
Theorem 2.19:

Let $R$ be AWN-injective ring with $lr(a^n) = l(r(a^{n-1}))$ for every $a \in N(R)$ and $a^n \neq 0$. Then $R$ is generalized n-regular.

Proof:

Suppose that $a \in N(R)$. Then there exists a positive integer $n$ such that $a^n \neq 0$ and $lr(a^n) = Ra^n \oplus X$ for some $X \leq R$. Since $lr(a^n) = l(r(a^{n-1}))$, then $lr(a^{n-1}) = Ra^n \oplus X$ and $a^{n-1} = da^n + x$ for some $d \in R$, $x \in X$. So $a^n = ada^n + ax$, $ax = a^n - ada^n \in Ra^n \cap X = 0$, $a^n = ada^n$.

This proves that $R$ is generalized n-regular.

Definition 2.20:

A ring $R$ is called right Quasi-Nduo ring if every right maximal right ideal is right Nduo.

Theorem 2.21:

Let $R$ be a right quasi N duo and every simple right $R$-module is AWN-injective. Then every element of $N(R)$ is strongly $\Pi$-regular.

Proof:

For any $0 \neq a \in N(R)$, we will show that there exists a positive integer $n$ such that $a^nR + r(a^n) = R$. Suppose not, then there exists a maximal right ideal $M$ of $R$ containing $a^nR + r(a^n)$. Since $R/M$ is AWN-injective, $l_{R/M}(r_R(a^n)) = (R/M)a^n + X_{a^n}$, $X_{a^n} \leq R/M$ and $a^n \neq 0$. Let $f : a^nR \to R/M$ be defined by $f(a^n) = r + M$. Since $a^nR + r(a^n) \subseteq M$, $f$ is well defined $R$-homomorphism. Thus there exists $c \in R$, $x \in X_{a^n}$ such that $1 + M = ca^n + M + x$, by Lemma (2.2), then $1 - ca^n + M = x \in (R/M)a^n \cap X_{a^n} = 0$. 
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1−ca^n ∈ M and ca^n ∈ M ( R is right N duo) and So 1∈ M , which is a contradiction.
Therefore a^nR+r(a^n) = R . In particular a^n x + y = 1 , x ∈ R , y ∈ r(a^n) , So a^n = a^{2n} + x .
Thus a is strongly Π-regular.

3- On Simple Singular AWN-injective Modules

In this section, we study of rings whose Simple singular right R-module are AWN-injective.
Also we give the relation between this rings and reduced rings.

A right MC2 ring R is called strongly right MC2 if R is also weakly reversible ring [12].

Now, the following result is given:

Proposition 3.1:

Let R be a ring whose every simple singular right R-module is AWN-injective. Then
Y(R) ∩ Z(R) = 0.

Proof:

If Y(R) ∩ Z(R) ≠ 0, then there exists 0 ≠ b ∈ Y(R) ∩ Z(R) such that b^2 = 0. We
claim that RbR+r(b) = R. Otherwise there exists a maximal essential right ideal M of R
containing RbR+r(b). So R/M is AWN-injective, and l_{r/M} r_k(b) = (R/M)b ⊕ X_b.

X_b ≤ R/M. Let f : bR → R/M be defined by f(br) = r + M. Note that f is a well
defined R-homomorphism. Then 1 + M = f(b) = cb + M + x , c ∈ R , x ∈ X_b ,
1−cb + M = x ∈ R/M ∩ X_b = 0 , 1−cb ∈ M. Since cb ∈ RbR ⊆ M , 1 ∈ M , which is a
contradiction. Therefore 1 = x + y , x ∈ RbR , y ∈ r(b) , and so b = bx. Since
RbR ⊆ Z(R) , x ∈ Z(R). Thus l(1−x) = 0 and so b = 0, which is a contradiction. This
show that Y(R) ∩ Z(R) = 0.
Theorem 3.2:

$R$ is a reduced ring if and only if $R$ is a strongly right MC2 ring whose simple singular right $R$-modules are AWN-injective.

Proof:

The necessity is evident.

Conversely: Let $a^2 = 0$. Suppose that $a \neq 0$. Then there exists a maximal right ideal $M$ of $R$ containing $r(a)$. First observe that $M$ is an essential right ideal of $R$. If not, then $M = r(e)$ for some $e \in ME_e$, (the set of all minimal idempotents elements of $R$). Since $R$ is strongly right MC2 ring, $R$ is a strongly min-right semi central ring, so we obtain $e$ is central in $R$. Using $a \in r(a)$, we get $ae = ea = 0$. Hence $e \in r(a) \subseteq M = r(e)$. Which is a contradiction. Therefore $M$ must be an essential right ideal of $R$. Thus $R/M$ is AWN-injective, and there exists a positive integer $n \geq 1$ such that $a^n \neq 0$ and $1_{R/M}r_r(a^n) = (R/M)a^n \oplus X_{a^n}$, $X_{a^n} \leq R/M$. Since $a^2 = 0$, then $n = 1$, and therefore $1_{R/M}r_r(a) = (R/M)a \oplus X_a$. Let $f : aR \rightarrow R/M$ defined by $f(ar) = r + M$. Note that $f$ is a well-defined $R$-homomorphism. Since $R/M$ is AWN-injective, there exists $c \in R$ such that $1 + M = f(a) = ca + M + x, x \in X_a$ (Lemma 2.2). So $1 - ca + M = x \in R/M \cap X_a = 0$. Since $a^2 = 0$, $aca \subseteq N^+(R)$ (the sum of all nil ideal) $\subseteq N(R)$. Hence $ca \in N(R)$ and so $1 - ca \in U(R)$ (the set of all invertible elements), which implies that $M = R$, which is a contradiction. Therefore $a = 0$, and $R$ is reduced.

Theorem 3.3:

Let $R$ be a NCI ring. If $R$ satisfies one of the following conditions, then $R$ is a reduced ring:

1- $R$ is a right $n$-weakly regular.
2- Every simple right $R$-modules is AWN-injective.

3- $R$ is right MC2 whose every simple singular right module is AWN-injective.

Proof:

If $N(R) \neq 0$, there exists $0 \neq I$ of $R$ contained in $N(R)$. Clearly, there exists $0 \neq b \in I$ such that $b^2 = 0$ and so there exists a maximal right ideal $M$ of $R$ containing $r(b)$.

If $R$ is right n-weakly regular, then $b = bc$ for some $c \in RbR$. Since $RbR \subseteq I \subseteq N(R)$, there exists a positive integer $n \geq 1$ such that $c^n = 0$. Hence $b = bc = ccb = ccccb = \ldots c^n b = 0$, which is a contradiction.

If $R/M$ is AWN-injective, then $l_{R/M} r(b^n) = (R/M)b^n \varoplus X_{b^n}$, $X_{b^n} \leq R/M$. Since $b^2 = 0$, then $l_{R/M} r(b) = (R/M)b \varoplus X_b$. Let $f : bR \rightarrow R/M$ be defined by $f(br) = r + M$.

Note $f$ is a well defined. So $1 + M = f(b) = cb + M + x$, $c \in R$, $x \in X_b$. $1 - cb + M = x \in R/M \cap X_b = 0$. $1 - cb \in M$.

Since $cb \in I \subseteq N(R)$, $1 - cb \in U(R)$, which implies that $M = R$, a contradiction.

If $M$ is not an essential right ideal of $R$, then $M = r(e)$ for some $e \in ME, (R)$. Clearly $eb = 0$. If $eRb \neq 0$, then $eRbR = eR$. But $eRbR \subseteq I \subseteq N(R)$, which is a contradiction, because $e \notin N(R)$. So $eRb = 0$. Therefore $M$ is essential, then $R/M$ is AWN-injective and $l_{R/M} r(b) = (R/M)b \varoplus X_b$. Hence by the same method as in the proof of (2), a contradiction. Therefore $R$ is reduced.

A ring $R$ is said to be NI if $N(R)$ forms an ideal of $R$. A ring $R$ is said to be 2-prim if $N(R) = P(R)$, where $P(R)$ is the prime radical of $R$. Clearly, every 2-prime ring is NI [9].
Theorem 3.4 :

Let $R$ a right MC2 ring whose every Simple singular right $R$-module is AWN-injective, then the following conditions are equivalent :

1- $R$ is reduced ring .
2- $R$ is 2-prime ring .
3- $R$ is NI ring .

Proof :

$1 \rightarrow 2 \rightarrow 3$ are obviously .

$(3) \rightarrow (1)$ Let $a^2 = 0$. Suppose $a \neq 0$. Then there exists a maximal right ideal $M$ of $R$ containing $r(a)$. If $M$ is not essential in $R$, then $M = r(e)$, where $e^2 = e \in R$ is a right minimal element. Hence $ea = 0$ because $a \in r(a)$. If $eRa \neq 0$, then $eRaR = eR$. Since $R$ is NI ring, then $N(R)$ is an ideal of $R$, so $eRaR \subseteq N(R)$ because $a \in N(R)$. Thus $e \in N(R)$, which is a contradiction. This show that $eRa = 0$. Hence $aRe = 0$ because $R$ is right MC2. Thus $e \in r(a) \subseteq r(e)$ which is also a contradiction. This implies that $M$ is essential in $R$, then $R/M$ is AWN-injective. By hypothesis. So $l_{R/M}r(a) = (R/M)a \oplus X_a$, $X_a \leq R/M$ ($a^2 = 0$, then $n=1$). Let $f : aR \rightarrow R/M$ be defined by $f(ar) = r + M$. Note that $f$ is well defined $R$-homomorphism. Then $1 + M = f(a) = ca + M + x$, $c \in R$, $x \in X_a$, $1 - ca + M = x \in R/M \cap X_a = 0$, $1 - ca \in M$. Since $ca \in N(R)$, $1 - ca$ is invertible. So $M = R$, which is a contradiction. This show that $a = 0$ and so $R$ is reduced .

Call a ring $R$ right GMC2 for any $a \in R$, any right minimal idempotent $e \in R$, $eRa = 0$ implies $aRe = 0$. Clearly, a right GMC2 ring is right MC2 . [12]
Lemma 3.5 [12]:

Let $R$ be a right GMC2 ring and if $a \in R$ is not a right weakly regular element, then every maximal right ideal $M$ of $R$ containing $RaR + r(a)$ must be essential in $R$.

Proposition 3.6:

Let $R$ be a right GMC2 ring and if every simple singular right $R$-module is AWN-injective, then for any $0 \neq a \in N(R)$ there exists a positive integer $n$ such that $a^n \neq 0$ and $RaR + r(a^n) = R$.

Proof:

Assume that $a^n \neq 0$, $a^{n+1} = 0$. If $a^n$ is a right weakly regular element, then we are done. Otherwise, by Lemma (3.5), there exists a maximal essential right ideal containing $Ra^nR + r(a^n)$. Thus $R/M$ is AWN-injective and $l_{R/M}r(a^n) = (R/M)a^n \oplus X_{a^n}$, $X_{a^n} \leq R/M$. Let $f : a^nR \to R/M$ be defined by $f(a^n r) = r + M$. Note that $f$ is a well-defined $R$-homomorphism. Then $1 + M = f(a^n) = da^n + M + x$, $d \in R$, $x \in X_{a^n}$, $1 - da^n + M = x \in R/M \cap X_{a^n} = 0$, $1 - da^n \in M$. Since $da^n \in Ra^nR \subseteq M$, $1 \in M$, which is a contradiction. Hence $R = Ra^nR + r(a^n) = RaR + r(a^n)$.

From Theorem (3.4) and proposition (3.6) we get:

Corollary 3.7:

Let $R$ be a right GMC2, NI ring, whose every simple singular right $R$-module is AWN-injective. Then $R$ is weakly regular ring.

Theorem 3.8:

If $R$ is strongly right MC2, then the following statements are equivalent:

Every right $R$-module is WN-injective.
Every right $R$-module is AWN-injective.
Every simple right $R$-module is AWN-injective.
Every simple singular right $R$-module is AWN-injective.
\( R \) is reduced.
\( R \) is n-regular .

**Proof** :

Obviously (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4), (5) \( \Rightarrow \) (6). And by [Theorem 3.2], (4) implies (5).

(6) \( \Rightarrow \) (1) Lemma (2.1).

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**REFERENCES**


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