ON THE THIRD HANKEL DETERMINANT FOR A SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

PRAVATI SAHOO

Abstract. Let \( A \) denote the class of all normalized analytic function \( f \) in the unit disc \( U \) of the form \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \). The object of this paper is to obtain a bound to the third Hankel determinant denoted by \( H_3(1) \) for a subclass of close-to-convex functions.

1. Introduction

Let \( \mathcal{A} \) denote the class of all analytic functions defined on the unit disc \( U = \{ z : |z| < 1 \} \) with the normalization condition \( f(0) = 0 = f'(0) - 1 \). So \( f \in \mathcal{A} \) has the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n.
\]

Let \( \mathcal{S} \) be the class of all functions \( f \in \mathcal{A} \) which are univalent in \( U \). Let \( \mathcal{P} \) denote the class of functions \( p(z) \), has the form

\[
p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,
\]

which are regular in the open unit disc \( U \) and satisfy the conditions \( p(0) = 1 \) and \( \Re p(z) > 0 \), for \( z \in U \). Here \( p(z) \) is called the Caratheodory function [5]. A function \( f \in \mathcal{A} \) is said to be starlike if it satisfies the condition \( \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \), for \( z \in U \).

1991 Mathematics Subject Classification. 30C45, 30C55.

Key words and phrases. Univalent functions, starlike, convex functions, close-to-convex functions, Fekete-Szegő inequality, Hankel determinant.

Copyright © Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

Received: Nov. 19, 2017 Accepted: Sept. 26, 2018.
and $S^*$, be the class of all starlike functions. Further, a function $f \in A$ is said to be close-to-convex if it satisfies the condition $Re\left\{zf'(z)/\phi(z)\right\} > 0$, for $z \in U$ and for a starlike function $\phi(z)$. $K$ be the class of all close-to-convex functions which was introduced by Kaplan [9]. In 1977, Chichra [4] introduced a new subclass of $K$ defined as follows:

**Definition 1.1.** ([4]) For $\alpha \geq 0$, a function $f \in A$ with $f(z)f'(z) \neq 0$ said to be alpha-close-to-convex function if for a starlike function $\phi(z)$, satisfies the condition

$$Re\left\{(1 - \alpha)zf'(z)\phi(z) + \alpha(zf'(z))'\phi'(z)\right\} > 0, \quad z \in U$$

and $C_\alpha$ be the class of all alpha-close-to-convex functions.

For $\alpha = 0$, $C_\alpha \equiv K$. For $\phi(z) = f(z)$, the class $C_\alpha$ is the class of alpha-starlike functions, which was introduced and studied by P.T. Mocanu [15], also studied in [17]. Chichra [4] proved that every alpha-close-to-convex function is close-to-convex. Also established the following theorem:

**Theorem 1.1.** ([4]) Let for $\alpha \geq 0$, $f \in C_\alpha$. Then

$$|a_2| \leq \frac{2 + \alpha}{1 + \alpha}; \quad |a_3| \leq \frac{9 + 23\alpha + 6\alpha^2}{3(1 + \alpha)(1 + 2\alpha)}; \quad |a_4| \leq \frac{4 + 22\alpha + 34\alpha^2 + 6\alpha^3}{4(1 + \alpha)(1 + 2\alpha)(1 + 3\alpha)}.$$

The inequalities are sharp.

Later in [1], Babalola derived the sharp upper bounds of the fifth coefficient of the functions in $C_\alpha$ as follows:

$$|a_5| \leq \frac{25 + 238\alpha + 755\alpha^2 + 902\alpha^3 + 120\alpha^4}{5(1 + \alpha)(1 + 2\alpha)(1 + 3\alpha)(1 + 4\alpha)}.$$

It is well known that, the coefficient problem in the univalent function theory ever attracts the function theorist. Closely related to the famous Bieberbach conjecture $|a_n| \leq n$ for $f \in S$, in 1933, Fekete-Szegö obtained the sharp bound for $|a_3 - \mu a_2^2|$, $\mu \in \mathbb{R}$ for $f \in S$. The functional $|a_3 - \mu a_2^2|$ is known as Fekete-Szegö functional. Many more functionals risen after it, each finding application in certain problems.
of geometric functions. For $\mu = 1$, a more general coefficient problem of this type, which is the Hankel determinant problem.

**Definition 1.2.** The $q$-th Hankel determinant of $f(z)$ for $q \geq 1$ and $n \geq 1$ is defined by Pommerenke [20] as

\[
H_q(n) = \begin{vmatrix}
a_n & a_{n+1} & \cdots & a_{n+q-1} \\
a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}
\end{vmatrix}.
\]

In the recent years a great deal of attention has been devoted for finding the estimates of Hankel determinants whose elements are the coefficients of the univalent functions for different specific values of $q$ and $n$. For example, Noonan and Thomas [18] studied about the second Hankel determinant of areally mean $p$-valent functions. Noor [19] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in $S$ with a bounded boundary. Ehrenborg [6] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [13]. It is interesting to note that, $H_2(1) = |a_3 - a_2^2|$, the Fekete-Szeg"{o} functional for $\mu = 1$.

The Hankel determinant in the case of $q = 2$ and $n = 2$, is known as the second Hankel determinant, given by

\[
H_2(2) = \begin{vmatrix}
a_2 & a_3 \\
a_3 & a_4
\end{vmatrix} = a_2a_4 - a_3^2.
\]

The bounds of $H_2(2)$ were obtained for various subclasses of univalent and multivalent analytic functions by many authors existed in the literature [3, 8, 11, 14, 16, 21, 22]. Similarly, the third Hankel determinant in the case of $q = 3$ and $n = 1$, denoted by
$H_3(1)$, is defined by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}. $$

For $f \in \mathcal{A}$, $a_1 = 1$, we have

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

and by applying triangle inequality, we obtain

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|. $$

Recently, Babalola [2], Bansal et al [3], Prajapat et al [23], Vamshee Krishna et al [11], have studied the third Hankel determinant and obtained the bounds of the determinants $|H_2(2)|$ and $|H_3(1)|$ for $\mathcal{K}$, the class of close-to-convex functions. Also in [24], Sahoo obtained the bounds of the determinants $|H_2(2)|$ and $|H_3(1)|$ for a subclass of $\alpha$-starlike functions. Motivated by the results obtained by Chichra [4], Babalola [2] and Prajapat et al [23], we obtain an upper bound to $|H_2(2)|$ and $|H_3(1)|$ for the function $f(z)$ in the class $\mathcal{C}_\alpha$.

2. Preliminary Results

The following lemmas are required to prove our main results.

**Lemma 2.1.** ([20] pp. 41). If $p(z) \in \mathcal{P}$, given by (1.2), then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $p_0(z) = \frac{1+z}{1-z}$.

**Lemma 2.2.** ([12]) Let $p(z) \in \mathcal{P}$, given by (1.2), then

$$2c_2 = c_1^2 + x(4 - c_1^2),$$

for some $x$, $|x| \leq 1$, and

$$4c_3 = c_1^2 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z,$$
for some $x$ and $z$ such that $|x| \leq 1$ and $|z| \leq 1$.

**Lemma 2.3.** [5] If $f \in S^*$ be given by (1.1), then $|a_n| \leq n$, $n = 2, 3, \ldots$. Equality holds for the rotations of the Koebe function $k(z) = \frac{z}{(1-z)^2}$.

**Lemma 2.4.** [10] If $f \in S^*$ be given by (1.1), then for any real number $\mu$, we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } 0 \leq \mu \leq \frac{1}{2}, \\ 1, & \text{if } \frac{1}{2} \leq \mu < 1, \\ 4 - 3\mu, & \text{if } \mu \geq 1, \end{cases}$$

**Lemma 2.5.** [7] If $f \in S^*$ be given by (1.1), then $|a_2 a_4 - a_3^2| \leq 1$. Equality holds for Koebe function.

**Lemma 2.6.** [2] If $f \in S^*$ be given by (1.1), then $|a_2 a_3 - a_4| \leq 2$. Equality holds for Koebe function.

### 3. Main Results

To obtain our result, we refer to the classical method initiated by Libera and Zlotkiewicz [12].

**Theorem 3.1.** For $0 \leq \alpha \leq 2/3$, let $f \in C_\alpha$. Then

$$|H_2(2)| = |a_2 a_4 - a_3^2| \leq \frac{85 + 3\alpha[247 + 509\alpha + 397\alpha^2 + 152\alpha^3 + 36\alpha^4]}{36(1 + \alpha)^2(1 + 2\alpha)^2(1 + 3\alpha)}.$$

**Proof.** Let $f(z)$ given by (1.1), be in the class $C_\alpha$. Then there exists an analytic function $p \in \mathcal{P}$ given by (1.2) and a starlike function $\phi(z) = z + \sum_{n=2}^{\infty} b_n z^n$, such that

$$(1 - \alpha)zf''(z)\phi'(z) + \alpha\phi(z)(zf'(z))' = p(z)\phi(z)\phi'(z).$$

On substituting power series expansion of $f(z)$, $p(z)$ and $\phi(z)$, and comparing the coefficient of $z^n$ on both sides we obtain

$$\sum_{k=0}^{n-1} (n-k)((n-2k-1)\alpha + k + 1)b_{k+1}a_{n-k} = \sum_{k=0}^{n-1} q_{k+1}b_{n-k}, \quad n \geq 2,$$
where \( a_1 = b_1 = q_1 = 1 \) and for \( k \geq 2 \),

\[
q_k = \sum_{j=0}^{k-1} (k-j)c_j b_{k-j}.
\]

Thus we have,

\[
q_2 = 2b_2 + c_1, \quad q_3 = 3b_3 + 2b_2c_1 + c_2, \quad q_4 = 4b_4 + 3b_3c_1 + 2b_2c_2 + c_3.
\]

On substituting the values of \( q_2, q_3, q_4 \) in (3.1), and comparing the coefficients of \( z^2, z^3 \) and \( z^4 \) we get

\[
2(1 + \alpha)a_2 = (1 + \alpha)b_2 + c_1
\]

\[
3(1 + \alpha)(1 + 2\alpha)a_3 = (1 + \alpha)(1 + 2\alpha)b_3 + (1 + 3\alpha)b_2c_1 + (1 + \alpha)c_2
\]

\[
4(1 + \alpha)(1 + 2\alpha)(1 + 3\alpha)a_4 = (1 + \alpha)(1 + 2\alpha)(1 + 3\alpha)b_4
\]

\[
+ (1 + 2\alpha)(1 + 5\alpha)b_3c_1 + (1 + \alpha)(1 + 5\alpha)b_2c_2
\]

\[
+ (1 + \alpha)(1 + 2\alpha)c_3 + \alpha(\alpha - 1)b_2^2c_1.
\]

Now

\[
|a_2a_4 - a_3^2| = \left| \left( \frac{b_2}{2} + \frac{c_1}{2(1 + \alpha)} \right) \left[ \frac{b_4}{4} + \left\{ \frac{1 + 5\alpha}{4(1 + \alpha)(1 + 3\alpha)} b_3
\right.
\]

\[
- \frac{\alpha(\alpha - 1)}{4(1 + \alpha)(1 + 2\alpha)(1 + 3\alpha)} b_2^2 \right] c_1 + \frac{1 + 5\alpha}{4(1 + 2\alpha)(1 + 3\alpha)} b_2c_2
\]

\[
+ \frac{1}{4(1 + 3\alpha)} c_3 - \left( \frac{b_3}{3} + \frac{1 + 3\alpha}{3(1 + \alpha)(1 + 2\alpha)} b_2c_1 + \frac{1}{3(1 + 2\alpha)} c_2 \right)^2 \right|
\]

\[
= \frac{1}{8} (b_2b_4 - b_3^2) + \frac{1}{8(1 + \alpha)} (b_4 - b_2b_3)c_1 + \frac{1 + 5\alpha}{8(1 + \alpha)^2(1 + 3\alpha)} \times
\]

\[
[b_3 - \mu_1 b_2^2]_1^2 + \frac{1 + 6\alpha}{36K(\alpha)} b_2b_3c_1 - \frac{\alpha(1 - \alpha)}{8K(\alpha)} b_2^2c_1 + \frac{1}{72} b_3^2
\]
\[
\begin{align*}
&-\frac{2}{9(1+2\alpha)}(b_3 - \mu_2b_2^2)c_2 - \frac{1}{9(1+2\alpha)^2}c_2^3 - \frac{7 + 33\alpha + 54\alpha^2}{72K(\alpha)(1+2\alpha)}b_2c_1c_2 \\
&+ \frac{(1+2\alpha)}{8K(\alpha)}c_1c_3 + \frac{1}{8(1+3\alpha)}b_2c_3,
\end{align*}
\]

where
\[
K(\alpha) = (1+\alpha)(1+2\alpha)(1+3\alpha),
\]

\[
\mu_1 = \frac{8 + 81\alpha + 208\alpha^2 + 192\alpha^3}{9(1+5\alpha)(1+2\alpha)^2}, \quad \mu_2 = \frac{9(1+5\alpha)}{16(1+3\alpha)}.
\]

Substituting the values of \(c_2\) and \(c_3\) from Lemma 2.2 in the equation (3.6), we have
\[
\begin{align*}
|a_{2a_4} - a_3^2| &= \left| \frac{1}{8}(b_2b_4 - b_3^2) + \frac{1}{8(1+\alpha)}(b_4 - b_2b_3)c_1 + \frac{1+5\alpha}{8(1+\alpha)^2(1+3\alpha)}[b_3 - \mu_1b_2^2]c_1^2 \\
&+ \frac{1+6\alpha}{36K(\alpha)}b_2bc_1 + \frac{\alpha(1-\alpha)}{8K(\alpha)}b_2c_1^2 + \frac{1}{72}b_3^2 - \frac{1}{9(1+2\alpha)}[b_3 - \mu_2b_2^2]x \\
&\times [c_1^2 + (4 - c_1^2)x] - \frac{1}{36(1+2\alpha)^2}[c_1^2 + (4 - c_1^2)x]^2 - \frac{7 + 33\alpha + 54\alpha^2}{144K(\alpha)(1+2\alpha)} \\
&b_2c_1[c_1^2 + (4 - c_1^2)x] + \frac{1}{32(1+\alpha)(1+3\alpha)}[(1+\alpha)b_2 + c_1]x \\
&[c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)x] \\
&= \left| \frac{1}{8}(b_2b_4 - b_3^2) + \frac{1}{8(1+\alpha)}(b_4 - b_2b_3)c_1 + \frac{1+6\alpha}{36K(\alpha)}b_2bc_1 - \frac{\alpha(1-\alpha)}{8K(\alpha)}b_2c_1^2 \\
&+ \frac{1+4\alpha + 12\alpha^2}{288(1+2\alpha)K(\alpha)}c_4 + \frac{1}{72}b_3^2 + \frac{1+5\alpha}{8(1+\alpha)^2(1+3\alpha)}[b_3 - \mu_1b_2^2]c_1^2 \\
&- \frac{1}{9(1+2\alpha)}[b_3 - \mu_2b_2^2]c_1^2 - \frac{1}{9(1+2\alpha)}[b_3 - \mu_2b_2^2](4 - c_1^2)^2x \\
&5 + 21\alpha + 36\alpha^2(1-\alpha) - \frac{b_2c_1^3 + 1+6\alpha + 9\alpha^2(1+2\alpha)}{72(1+2\alpha)K(\alpha)}b_2c_1(4 - c_1^2)x \right|
\end{align*}
\]
\begin{align}
&\frac{1 + 4\alpha + 12\alpha^2}{288(1 + 2\alpha)K(\alpha)}c^2(4 - c^2)x - \frac{c_1^2(4 - c_1^2)x^2}{32(1 + \alpha)(1 + 3\alpha)} - \frac{(4 - c_1^2)^2x^2}{36(1 + 2\alpha)^2} \\
&\frac{c_1(4 - c_1^2)x^2}{32(1 + 3\alpha)}b_2 + \frac{1 + 2\alpha}{16K(\alpha)}(1 + \alpha)b_2 + c_1[(4 - c_1^2)(1 - |x|^2)].
\end{align}

By Lemma 2.1, we have $|c_1| \leq 2$. For convenience of notation, we take $c_1 = c$ and we may assume without loss of generality that $c \in [0, 2]$. Applying the triangle inequality with $|x| = \mu$ and using Lemma 2.3, Lemma 2.4, Lemma 2.5 and Lemma 2.6, we obtain from the above inequality

\begin{align}
|a_2a_4 - a_3^2| &\leq \left[ \frac{1}{8} |b_2b_4 - b_3^2| + \frac{|b_4 - b_2b_3|c}{8(1 + \alpha)} + \frac{(1 + 6\alpha)|b_2||b_3|c}{36K(\alpha)} + \frac{\alpha(1 - \alpha)|b_2|^3c}{8K(\alpha)} \\
+ \frac{1 + 4\alpha + 12\alpha^2}{288(1 + 2\alpha)K(\alpha)}c^4 + \frac{1}{72} |b_3|^2 + \frac{1 + 5\alpha}{8(1 + \alpha)^2(1 + 3\alpha)} |b_3 - \mu_1b_2|^2c^2 \\
+ \frac{1}{9(1 + 2\alpha)} |b_3 - \mu_2b_2|^2c^2 + \frac{1}{9(1 + 2\alpha)} |b_3 - \mu_2b_2|(|4 - c^2|^2\mu \\
+ \frac{5 + 21\alpha + 36\alpha^2(1 - \alpha)}{288(1 + 2\alpha)K(\alpha)} |b_2|c^3 + \frac{1 + 6\alpha + 9\alpha^2(1 + 2\alpha)}{72(1 + 2\alpha)K(\alpha)} |b_2|c(4 - c^2)\mu \\
+ \frac{1 + 4\alpha + 12\alpha^2}{144(1 + 2\alpha)K(\alpha)} c^2(4 - c^2)\mu + \frac{c^2(4 - c^2)\mu^2}{32(1 + \alpha)(1 + 3\alpha)} + \frac{(4 - c^2)^2\mu^2}{36(1 + 2\alpha)^2} \\
+ \frac{c(4 - c^2)\mu^2}{32(1 + 3\alpha)} |b_2| + \frac{1 + 2\alpha}{16K(\alpha)} [(1 + \alpha)|b_2| + c](4 - c^2)(1 - \mu^2)]
\end{align}

\begin{align}
&\leq \left[ \frac{3(1 + \alpha)}{4(1 + 3\alpha)} + \frac{A_1(\alpha)}{12k(\alpha)}c + \frac{A_2(\alpha)}{72(1 + 2\alpha)k(\alpha)}c^2 - \frac{A_3(\alpha)}{144(1 + 2\alpha)k(\alpha)}c^3 \\
+ \frac{A_4(\alpha)}{288(1 + 2\alpha)k(\alpha)}c^4 + \frac{(4 - c^2)\mu}{144(1 + 2\alpha)K(\alpha)} [B_1(\alpha) + B_2(\alpha)c + B_3(\alpha)c^2] \\
+ \frac{(4 - c^2)\mu^2}{288(1 + 2\alpha)K(\alpha)} [D_1(\alpha) + D_2(\alpha)c + D_3(\alpha)c^2] \right] \\
= &\quad F_1(c, \mu),
\end{align}
where

\begin{align}
(3.11) \quad & A_1(\alpha) = 8 + 45\alpha + 6\alpha^2, \quad A_2(\alpha) = 8 + 67\alpha + 101\alpha^2 + 6\alpha^3, \quad A_3(\alpha) = 4 + 15\alpha + 36\alpha^3, \\
(3.12) \quad & B_1(\alpha) = 16K(\alpha), \quad B_2(\alpha) = 4(1 + 6\alpha + 9\alpha^2 + 18\alpha^3), \\
(3.13) \quad & B_3(\alpha) = D_3(\alpha) = A_4(\alpha) = 1 + 4\alpha + 12\alpha^2 \\
(3.14) \quad & D_1(\alpha) = -4(1 + \alpha)(1 + 12\alpha + 36\alpha^2), \quad D_2(\alpha) = 18\alpha(1 + 2\alpha)^2, \\
\end{align}

and \(K(\alpha)\) defined by (3.7).

Differentiating \(F_1(c, \mu)\) with respect to \(\mu\), we get

\[
\frac{\partial F_1}{\partial \mu} = \frac{(4 - c^2)}{144(1 + 2\alpha)K(\alpha)} \left[ 4(1 - \mu)(1 + \alpha)(1 + 12\alpha + 36\alpha^2) + 4(1 + \alpha)(3 + 8\alpha - 12\alpha^2) \right. \\
+ \left. 2\{2 + 3(4 + 3\mu)\alpha + 18(1 + 2\mu)\alpha^2(1 + \alpha) + 36(1 + \mu)\alpha^3\}c + (1 + \mu)(1 + 2\alpha)^2c^2 \right],
\]

which shows that \(\frac{\partial F_1}{\partial \mu} > 0\) for \(0 \leq \mu \leq 1\) and \(0 \leq \alpha < 1\). Therefore \(F_1(c, \mu)\) is an increasing function of \(\mu\) for \(0 \leq \mu \leq 1\) and for any fixed \(c\) with \(c \in [0, 2]\). So it attains maximum at \(\mu = 1\). Thus

\begin{align}
(3.15) \quad & \max_{0 \leq \mu \leq 1} F_1(c, \mu) = F_1(c, 1) = G_1(c) \ (say). \\
\end{align}

Therefore from (3.10) and (3.15) we have

\[
G_1(c) = \frac{3(1 + \alpha)}{4(1 + 3\alpha)} + \frac{A_1(\alpha)}{12k(\alpha)}c + \frac{A_2(\alpha)}{72(1 + 2\alpha)k(\alpha)}c^2 - \frac{A_3(\alpha)}{144(1 + 2\alpha)k(\alpha)}c^3 \\
+ \frac{B_3(\alpha)}{288(1 + 2\alpha)k(\alpha)}c^4 + \frac{(4 - c^2)}{144(1 + 2\alpha)K(\alpha)} \times \\
[(D_1(\alpha) + 2B_1(\alpha)) + (D_2(\alpha) + 2B_2(\alpha))c + 2B_3(\alpha)c^2],
\]

(3.16)
where $A_1(\alpha), A_2(\alpha), A_3(\alpha), B_1(\alpha), B_2(\alpha), B_3(\alpha)$ and $D_1(\alpha), D_2(\alpha)$ defined in (3.11), (3.12), (3.13) and (3.14) respectively, and $K(\alpha)$ defined by (3.7). On differentiating $G_1(c)$ with respect to $c$, we get

$$G_1'(c) = \frac{1 + 3\alpha}{144K^2(\alpha)} \left[ 4(4 - c^2)(1 + \alpha) \{ (1 + 4\alpha + 12\alpha^2)c + 6(1 + 6\alpha + 9\alpha^2 + 18\alpha^3) \} + 3\alpha(7 + 32\alpha + 40\alpha^2)c + 8(1 + \alpha)(2 + 36\alpha + 72\alpha^2 - 153\alpha^3) \right],$$

which shows that $G_1'(c) > 0$ for $0 \leq c \leq 2$. So $G_1(c)$ is increasing function of $c$, hence it will attains maximum at $c = 2$. Therefore

$$\max_{0 \leq c \leq 2} G_1(c) = G_1(2) = \frac{85 + 3\alpha[247 + 509\alpha + 397\alpha^2 + 152\alpha^3 + 36\alpha^4]}{36(1 + \alpha)^2(1 + 2\alpha)^2(1 + 3\alpha)}.$$

Hence the upper bound on $|a_2a_4 - a_3^2|$ can be obtained by setting $\mu = 1$ and $c = 2$ in (3.10). Hence the desired result follows from (3.10) and (3.17). \hfill \Box

For $\alpha = 0$, the result was proved in [23].

**Theorem 3.2.** For $0 \leq \alpha \leq \frac{1}{2}$, let $f \in \mathcal{C}_\alpha$. Then

$$|a_2a_3 - a_4| \leq \frac{9 + 52\alpha + 83\alpha^2 + 37\alpha^3 + 18\alpha^4}{3(1 + \alpha)^2(1 + 2\alpha)(1 + 3\alpha)}.$$

**Proof.** Let $f(z)$ given by (1.1), be in the class $\mathcal{C}_\alpha$. Then substituting the values of $a_2, a_3$ and $a_4$ from (3.3), (3.4) and (3.5) in $|a_2a_3 - a_4|$, we have

$$|a_2a_3 - a_4| = \left| \left( \frac{b_2}{2} + \frac{c_1}{2(1 + \alpha)} \right) \left( \frac{b_3}{3} + \frac{1 + 3\alpha}{3(1 + \alpha)(1 + 2\alpha)}b_2c_1 + \frac{1}{3(1 + 2\alpha)}c_2 \right) \right|$$
Substituting the values of \( |s| \) and \( \alpha \), we have,

\[
\begin{aligned}
By the Lemma 2.1, we have \\
(3.19) \quad \mu = \frac{2 + 15\alpha(1 + \alpha)}{(1 + 9\alpha)(1 + 2\alpha)} \\
\end{aligned}
\]

where

\[
(3.18) \quad |2(1 + 3\alpha)c_1 - E_1(\alpha)b_2|c_2 + \frac{(1 + 3\alpha)^2}{6(1 + \alpha)K(\alpha)}b_2c_2^2 - \frac{1}{4(1 + 3\alpha)c_3},
\]

and \( K(\alpha) \) defined by (3.7).

Substituting the values of \( c_2 \) and \( c_3 \) from Lemma 2.2 in the equation (3.18), and on simplification we have,

\[
|a_2a_3 - a_4| = \left| \frac{b_2b_3}{4} - \frac{b_2b_3}{12} - \frac{E_2(\alpha)(b_3 - \mu_3b_2^2)c_1}{12K(\alpha)} + \frac{(3 + 13\alpha + 17\alpha^2 - 9\alpha^3)b_2c_1^2}{24(1 + \alpha)K(\alpha)} \\
- \frac{(1 + 3\alpha - 6\alpha^2)c_1^2}{48K(\alpha)} - \frac{(1 + 3\alpha + 6\alpha^2)c_1(4 - c_1^2)x}{24K(\alpha)} - \frac{E_1(\alpha)(4 - c_1^2)b_2x}{24K(\alpha)} \\
+ \frac{(4 - c_1^2)c_4x^2}{16(1 + 3\alpha)} - \frac{(4 - c_1^2)(1 - |x|^2)z}{8(1 + 3\alpha)} \right|.
\]

By the Lemma 2.1, we have \( |c_1| \leq 2 \). For convenience of notation, we take \( c_1 = c \) and we may assume without loss of generality that \( c \in [0, 2] \). Applying the triangle inequality with \( |x| = \mu \) and using Lemma 2.3, Lemma 2.4 and Lemma 2.6, we obtain

\[
|a_2a_3 - a_4| \leq \frac{|b_2b_3 - b_4|}{4} + \frac{|b_2||b_3|}{12} + \frac{E_2(\alpha)}{12K(\alpha)}|b_3 - \mu_3b_2^2|c + \frac{3 + 13\alpha + 17\alpha^2 - 9\alpha^3}{24(1 + \alpha)K(\alpha)}
\]
\[ \frac{\partial F_2}{\partial c} = \frac{(5 + 27\alpha + 6\alpha^2) - (1 + 3\alpha + 6\alpha^2)\mu c^2}{12K(\alpha)} + \frac{(3 - \mu)(1 + 5\alpha) + 24\alpha^2(1 - \alpha) - 2(1 + \alpha)(1 + 9\alpha)\mu}{12(1 + \alpha)K(\alpha)} + \frac{(4 - c^2)(1 + 3\alpha + 6\alpha^2)}{24K(\alpha)} + \frac{1 + 3\alpha(1 - 2\alpha)}{26K(\alpha)}c^2 + \frac{(2 - c)(2 + 3\epsilon)}{16(1 + 3\alpha)}\mu, \]

which shows that \( \frac{\partial F_2}{\partial c} > 0 \) for \( 0 \leq c \leq 2 \). So \( F_2(c, \mu) \) is increasing function of \( c \), hence it will attains maximum at \( c = 2 \). Therefore

\[ \max_{0 \leq c \leq 2} F_2(c, \mu) = F_2(2, \mu) = G_2(\mu) \text{ (say)}. \]

From (3.21) and (3.22), we get

\[ G_2(\mu) = \frac{3(1 + 2\alpha)}{2(1 + 3\alpha)} + \frac{5 + 27\alpha + 6\alpha^2}{6K(\alpha)} + \frac{3 + 14\alpha + 19\alpha^2 - 24\alpha^3}{6(1 + \alpha)K(\alpha)} + \frac{1 + 3\alpha(1 - 2\alpha)}{6K(\alpha)}, \]

which is independent of \( \mu \). Hence the sharp upper bound of the functional \( |a_2a_3 - a_4| \) is \( F_2(2, \mu) = G_2(\mu) \). Thus the desired result follows from (3.21) and (3.23). \( \square \)

**Theorem 3.3.** For \( 0 \leq \alpha \leq \frac{1}{2} \), let \( f \in C_\alpha \). Then

\[ |a_3 - a_2^2| \leq \frac{4(1 + 2\alpha + 4\alpha^2 + 2\alpha^3)}{3(1 + 2\alpha)(1 + 2\alpha + 4\alpha^2)}. \]

**Proof.** The proof is similar to the proof of the Theorem 3.1 and Theorem 3.2. \( \square \)
**Theorem 3.4.** If for $0 \leq \alpha \leq \frac{1}{2}$, $f \in C_\alpha$, then

$$H_3(1) \leq \frac{1}{3(1+2\alpha)K(\alpha)} \left[ \frac{P_1(\alpha)}{12(1+\alpha)^2(1+2\alpha)} + \frac{P_2(\alpha)}{(1+\alpha)^2(1+3\alpha)} + \frac{P_3(\alpha)}{5(1+4\alpha)(1+2\alpha+4\alpha^2)} \right],$$

where $K(\alpha)$ defined by (3.7) and

- $P_1(\alpha) = [9 + 23\alpha + 6\alpha^2][85 + 3(247 + 509\alpha + 397\alpha^2 + 152\alpha^3 + 36\alpha^4)],$
- $P_2(\alpha) = [2 + 11\alpha + 17\alpha^2 + 3\alpha^3][9 + 52\alpha + 83\alpha^2 + 37\alpha^3 + 18\alpha^4],$
- $P_3(\alpha) = 4[1 + 2\alpha + 4\alpha^2 + 2\alpha^3][25 + 238\alpha + 755\alpha^2 + 902\alpha^3 + 120\alpha^4].$

**Proof.** Let for $0 \leq \alpha \leq \frac{1}{2}$, $f \in C_\alpha$. Then from (1.6) we have,

$$|H_3(1)| \leq |a_3||a_2a_4 - a_2^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|.$$

By using the bounds of $|a_2a_4 - a_2^2|$, $|a_2a_3 - a_4|$, $|a_3 - a_2^2|$ from Theorem 3.1, Theorem 3.2, Theorem 3.3 respectively, and the bounds $|a_3|$, $|a_4|$ due to Chichra[4], and the bound of $|a_5|$ due to Babalola[1] we get the desired result. \(\square\)

**Remark:** Finding the function for which the upper bound for $|H_2(2)|$ and $|H_3(1)|$ are to be sharp is an open problem. The nature of the function to be maximised much more complicated even for $\alpha = 0$, that is, for the functions in $K$.

**Acknowledgement**

I would like to thank the referees for their valuable suggestions and comments which improved the paper.
REFERENCES


Department of Mathematics, Banaras Hindu University,
Banaras 221 005, India

E-mail address: pravatis@yahoo.co.in