BAER GAMMA RINGS WITH INVOLUTIONS

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Abstract. The concept of involution in Γ-rings is introduced and with the help of involutions, we obtain some characterizations of Baer Γ-rings.

1. Introduction

As a generalization of rings, the concept of Γ-rings was first introduced by N. Nobusawa [6]. After words Barnes [1] generalized the notion of Nobusawa’s Γ-rings and gave a new definition of a Γ-ring. Now a days, Γ-rings means the Γ-rings in the sense of Barnes [1] where other Γ-rings are known as N -rings i.e., gamma rings in the sense of Nobusawa. Many Mathematicians worked on Γ-rings and obtained some fruitful results that are a generalization of many classical ring theories. In the Book ” Rings with operators ” Kaplansky [3] worked on Baer rings and obtained various results relating to involution and Baer rings. Paul and Sabur [9] worked on Lie and Jordan structures in simple Γ-rings and generalized some results of classical rings into Γ-rings. Paul and Sabur [10] also worked on Baer Gamma rings and obtained some characterizations of this Γ-ring.

In this paper, we introduce the notion of an involution in Γ-rings and generalize...
some results of classical Baer rings into gamma Baer rings with the help of the new concept of an involution. In [10], an example of a Baer gamma ring is given

2. Preliminaries

Definition 2.1. Gamma Ring: Let $M$ and $\Gamma$ be two additive abelian groups. Suppose that there is a mapping from $M \times \Gamma \times M \rightarrow M$ (sending $(x, \alpha, y)$ into $x\alpha y$) such that

1. $(x + y)\alpha z = x\alpha z + y\alpha z, x(\alpha + \beta)z = x\alpha z + x\beta z, x\alpha(y + z) = x\alpha y + x\alpha z$
2. $(x\alpha y)\beta z = x\alpha(y\beta z)$, where $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then $M$ is called a $\Gamma$-ring in the sense of Barnes [1].

Definition 2.2. Sub $\Gamma$-ring: Let $M$ be a $\Gamma$-ring. A non-empty subset $S$ of a $\Gamma$-ring $M$ is a sub $\Gamma$-ring of $M$ if $a, b \in S$, then $a - b \in S$ and $a\gamma b \in S, \forall \gamma \in \Gamma$.

Definition 2.3. Ideal of $\Gamma$-rings: A subset $A$ of the $\Gamma$-ring $M$ is a left (right) ideal of $M$ if $A$ is an additive subgroup of $M$ and $M\Gamma A = \{coa : c \in M, \alpha \in \Gamma, a \in A\}$ ($A\Gamma M$) is contained in $A$. If $A$ is both a left and a right ideal of $M$, then we say that $A$ is an ideal or two-sided ideal of $M$. If $A$ and $B$ are both left (respectively right or two-sided) ideals of $M$, then $A + B = \{a + b : a \in A, b \in B\}$ is clearly a left (respectively right or two-sided) ideal, called the sum of $A$ and $B$. We can say every finite sum of left (respectively right or two-sided) ideal of a $\Gamma$-ring is also a left (respectively right or two-sided) ideal.

It is clear that the intersection of any number of left (respectively right or two-sided) ideal of $M$ is also a left (respectively right or two-sided) ideal of $M$. If $A$ is a left ideal of $M$, $B$ is a right ideal of $M$ and $S$ is any non-empty subset of $M$, then the set, $A\Gamma S = \{\sum_{i=1}^{n} a_i \gamma s_i : a_i \in A, \gamma \in \Gamma, s_i \in S, n$ is a positive integer$\}$ is a left ideal of $M$ and $S\Gamma B$ is a right ideal of $M$. $A\Gamma B$ is a two-sided ideal of $M$. If $a \in M$, then the principal ideal generated by $a$ denoted by $<a>$ is the intersection
of all ideals containing $a$ and is the set of all finite sum of elements of the form 
\[ na + x\alpha a + a\beta y + u\gamma a \mu v, \]
where $n$ is an integer, $x, y, u, v$ are elements of $M$ and 
$\alpha, \beta, \gamma, \mu$ are elements of $\Gamma$. This is the smallest ideal generated by $a$. Let $a \in M$. 
The smallest left (right) ideal generated by $a$ is called the principal left (right) ideal 
\[ < a \mid (\mid a \mid). \]

**Definition 2.4. Unity element of a $\Gamma$-ring:** Let $M$ be a $\Gamma$-ring. $M$ is called a 
$\Gamma$-ring with unity if there exists an element $e \in M$ such that $a\gamma e = e\gamma a = a$ for all 
a $\in M$ and some $\gamma \in \Gamma$. We shall frequently denote $e$ by 1 and when $M$ is a $\Gamma$-ring 
with unity, we shall often write $1 \in M$. Note that not all $\Gamma$-rings have an unity. 
When a $\Gamma$-ring has an unity, then the unity is unique.

**Definition 2.5. Nilpotent element:** Let $M$ be a $\Gamma$-ring. An element $x$ of $M$ is 
called nilpotent if for some $\gamma \in \Gamma$, there exists a positive integer $n = n(\gamma)$ such that 
\[ (x\gamma)^n x = (x\gamma x\gamma...\gamma x\gamma ) x = 0. \]

**Definition 2.6. Nil ideal:** An ideal $A$ of a $\Gamma$-ring $M$ is a nil ideal if every element 
of $A$ is nilpotent that is, for all $x \in A$ and some $\gamma \in \Gamma$, 
\[ (x\gamma)^n x = (x\gamma x\gamma...\gamma x\gamma ) x = 0, \]
where $n$ depends on the particular element $x$ of $A$.

**Definition 2.7. Nilpotent ideal:** An ideal $A$ of a $\Gamma$-ring $M$ is called nilpotent if 
\[ (A\gamma)^n A = (A\gamma A\gamma...\gamma A\gamma ) A = 0, \]
where $n$ is the least positive integer.

**Definition 2.8. Annihilator of a subset of a $\Gamma$-ring:** Let $M$ be a $\Gamma$-ring. Let 
$S$ be a subset of $M$. Then the left annihilator $l(S)$ of $S$ is defined by 
$L(S) = \{ m \in M : m\gamma S = 0 \text{ for every } \gamma \in \Gamma \}$, whereas the right annihilator $r(S)$ is defined by 
\[ R(S) = \{ m \in M : S\gamma m = 0 \text{ for every } \gamma \in \Gamma \}. \]

**Definition 2.9. Idempotent element:** Let $M$ be a $\Gamma$-ring. An element $e$ of $M$ is 
called idempotent if $e\gamma e = e \neq 0$ for some $\gamma \in \Gamma$. 
Definition 2.10. Centre of a $\Gamma$-ring: Let $M$ be a $\Gamma$-ring. The centre of $M$, written as $Z$, is the set of those elements in $M$ that commute with every element in $M$, that is, $Z = \{ m \in M : m\gamma x = x\gamma m \text{ for all } x \in M \text{ and } \gamma \in \Gamma \}$.

Definition 2.11. $\Gamma M$-homomorphism: Let $M$ be a $\Gamma$-ring. Let $A$ and $B$ be the left ideals of $M$. A $\Gamma M$-homomorphism is a function $\phi : A \to B$ such that

(i) $\phi(x + y) = \phi(x) + \phi(y)$ for all $x, y \in A$
(ii) $(m\gamma x) = m\gamma(x)$ for all $x \in A, m \in M$ and $\gamma \in \Gamma$.

In case, $A$ and $B$ are right ideals, then (i) and (ii) become

(i) $\phi(x + y) = \phi(x) + \phi(y)$ for all $x, y \in A$
(ii) $\phi(x\gamma m) = \phi(x)\gamma m$ for all $x \in A, m \in M$ and $\gamma \in \Gamma$.

Definition 2.12. $\Gamma M$-isomorphism: Let $M$ be a $\Gamma$-ring. Let $A$ and $B$ be two left ideals of $M$. Let $\phi : A \to B$ be a $\Gamma M$-homomorphism from $A$ into $B$. We call $\phi$, a $\Gamma M$-isomorphism, if $\phi$ is one-one and onto. We say that $A$ and $B$ are $\Gamma M$-isomorphic and we write $A \cong B$.

Definition 2.13. Baer $\Gamma$-ring: A $\Gamma$-ring $M$ is called a Baer $\Gamma$-ring if the right annihilator of every non-empty subset of $M$ is generated by an idempotent element of $M$.

3. Baer Gamma Rings with Involutions

Definition 3.1. Let $M$ be a $\Gamma$-ring. A mapping $I : M \to M$ is called an involution if

(i) $I(a + b) = I(a) + I(b)$, (ii) $I(ab) = I(b)\alpha I(a)$ and (iii) $I^2(a) = a$, for all $a, b \in M, \alpha \in \Gamma$.

Example 3.1. Let $R$ be an associative ring with 1 having an involution $\ast$. Let $M = M1.2(R)$ and $\Gamma = \{ \begin{pmatrix} n_1,1 \\ n_2,1 \end{pmatrix} : n_1, n_2 \in \mathbb{Z} \}$. Then $M$ is a $\Gamma$-ring. Define $I : M \to M$ by $I((a, b)) = (a\ast, b\ast)$. Then it is clear that $I$ is an involution on $M$. 
We know that if \( e \) is the idempotent elements of a \( \Gamma \)-ring \( M \), then \( M e \) and \( e M \) are respectively left ideal and right ideal of \( M \), which is shown in [7].

**Theorem 3.1.** Let \( e \) and \( f \) be idempotents in a \( \Gamma \)-ring \( M \). The following are equivalent:

1. \( e \Gamma M, f \Gamma M \) are \( \Gamma M \)-isomorphism
2. \( M e, M f \) are \( \Gamma M \)-isomorphism
3. There exist elements \( x \in e \Gamma M, y \in f \Gamma M \) with \( x \alpha y = e, y \alpha x = f, \alpha \in \Gamma \).

**Proof.** Since condition (3) is left-right symmetric it will suffice to identify (2) and (3). (3) implies (2). Map \( M e \) to \( M f \) by right multiplication by \( x \), \( M f \) to \( M e \) by right multiplication by \( y \). The product both ways to clearly the identity. (2) implies (3). Let \( \phi \) be the map from \( M e \) to \( M f \) and set \( \phi(e) = x \).

Then since \( \phi \) is a \( \Gamma M \)-homomorphism map we have \( \phi(a) = \phi(a \alpha e) = a \alpha \phi(e) = a \alpha x \) for any \( a \in M e \) and \( \alpha \in \Gamma \) i.e., \( \phi \) is a right multiplication by \( x \). In particular \( x = \phi(e) = e \alpha x \) and \( x \in e \Gamma M f \). Similarly the map from \( M f \Gamma \) to \( M e \) is right multiplication by an element \( y \in f \Gamma M e \). Evidently \( x \alpha y = e, y \alpha x = f, \alpha \in \Gamma \). \( \Box \)

**Definition 3.2.** Idempotents \( e, f \) in a \( \Gamma \)-ring \( M \) are equivalent, written \( e \sim f \) if they satisfy (and hence all) of the conditions in theorem 3.3. Note that a Baer \( \Gamma \)-ring is finite if and only whenever \( e \sim 1 \), then \( e = 1 \).

**Definition 3.3.** An element of a \( \Gamma \)-ring \( M \) with involution \( I \) is called self-adjoint if \( I(x) = I \). A projection is a self-adjoint idempotent. A subset \( S \) is self-adjoint if \( x \in S \) implies \( I(x) \in S \). A Baer \( \Gamma \)-ring with involution \( I \) is a \( \Gamma \)-ring with involution \( I \) such that for any subset \( S, R(S) = e \Gamma M \) with \( e \) a projection.

By applying the involution we get that in a Baer \( \Gamma \)-ring with involution \( I \) the left annihilator of any subset is like wise generated by a projection. In particular, a Baer \( \Gamma \)-ring with involution \( I \) is a Baer \( \Gamma \)-ring. The projection of \( e \) generating \( e \Gamma M \) is unique. For if \( e \Gamma M = f \Gamma M \) with \( e \) and \( f \) projections, we find \( e = f \alpha e, \alpha \in \Gamma \).
$\Gamma, f = e\alpha f = I(e\alpha f) = I(f)\alpha I(e) = f\alpha e = I(f\alpha e) = I(e) = e$. Because of this uniqueness, we can call $e$ the right-annihilating projection of the subset $S$ of $M$.

Even more useful is $g = 1 - e$ which we shall call the right projection of $S$. Then $so g = so(1 - e) = so1 - so e = s$ for all $s \in S$ and $\alpha \in \Gamma$ is the smallest such projection. If $f$ is an idempotent in a Baer $\Gamma$-ring with involution $I$ and $e$ is its right projection, we readily see that $e \sim f$.

**Theorem 3.2.** Let $e$ and $f$ be idempotents in a Baer $\Gamma$-ring with involution $I, f \in e\Gamma M\Gamma e$. Let $g$ and $h$ be the right projections of $e$ and $f$. Then $e - f \sim g - h$.

**Proof.** Noting that $g \geq h, e\gamma g = e, g\gamma e = g, f\gamma h = f, h\gamma f = f, \gamma \in \Gamma$, we verify directly that $e - e\gamma h$ and $g - g\gamma f$ implement an equivalence of $e - g$ and $f - h$. □

**Definition 3.4.** For projection $e, f$ in a $\Gamma$-ring with involution $I$ write $e \leq f$ in case $e = e\gamma f, \gamma \in \Gamma$ (which is equivalent to $e = f\gamma e$). One readily sees that this relation makes the projections into a partially ordered set.

**Theorem 3.3.** The projections in a Baer $\Gamma$-ring with involution $I$ form a complete lattice.

**Proof.** Given a family $\{e_i\}$ of projections, let $e$ be their right projection. One readily sees that $e$ is the least upper bound (LUB) of $e_i$'s. Dually, there is a greatest lower bound (GLB). Hence the theorem is proved. □

**Definition 3.5.** Let $M$ be a Baer $\Gamma$-ring with involution $I$. $B$ is a sub $\Gamma$-ring of $M$.

We say that $B$ is a Baer sub $\Gamma$-ring with involution $I$ of $M$ if

1. $B$ is a self adjoint sub $\Gamma$-ring
2. If $S \subset B$ and $e$ is the right annihilating projection of $S(\text{in} M)$, then $e \in B$.

If $B$ is a Baer sub $\Gamma$-ring with involution $I$, then $B$ is itself obviously a Baer $\Gamma$-ring with involution $I$. Its unity element is the same as that of $M$ (take the annihilator of
0). The lattice of projections in $B$ is a complete sub lattice of that of $M$. If $M$ is a Baer $\Gamma$-ring with involution $I$ and $e$ is a projection in $M$, the projections of $e\Gamma M \Gamma e$ are the projections of $f \in M$ with $f \leq e$. It follows easily that $e\Gamma M \Gamma e$ is a Baer $\Gamma$-ring with involution $I$ and that a family of projections in $e\Gamma M \Gamma e$ has the same LUB whatever computed in $e\Gamma M \Gamma e$ or in $M$.

**Theorem 3.4.** Let $M$ be a Baer $\Gamma$-ring with involution $I$ and $S$ be a self-adjoint subset of $M$. Let $T$ be the commuting $\Gamma$-ring of $S$. Then $T$ is a Baer sub $\Gamma$-ring with involution $I$ of $M$.

**Proof.** Since $S$ is self-adjoint, the sub $\Gamma$-ring $T$ is also self-adjoint. Given $V \subset T$, write $R(V) = e\Gamma M$ (this is the annihilator in $M$ of course). We must show that $e$ lies in $T$. Thus given $V$, we have to prove $e\gamma s = s\gamma e$, $\gamma \in \Gamma$. Given $s \in S$, we have $s\gamma v = v\gamma s$ and $v\gamma e = 0$, then $v\gamma (1 - e)\gamma s\gamma e = v\gamma s\gamma e - v\gamma e\gamma s\gamma e = v\gamma s\gamma e - 0 = v\gamma s\gamma e = s\gamma v\gamma e = s\gamma 0 = 0$. Since $v$ is arbitrary in $V$, $(1 - e)\gamma s\gamma e \in e\Gamma M$. Hence $(1 - e)\gamma s\gamma e = 0$. Thus $s\gamma e = e\gamma s\gamma e$. Apply involution $I$, we have $I(sve) = I(e\gamma s\gamma e)$. This implies that $I(e)\gamma I(s) = I(e)\gamma I(s)\gamma I(e)$. So $e\gamma s = e\gamma s\gamma e$. Hence $s\gamma e = e\gamma s$. □

**Corollary 3.1.** The center of a Baer $\Gamma$-ring with involution $I$ is a Baer sub $\Gamma$-ring with involution $I$.

**Theorem 3.5.** In a Baer $\Gamma$-ring $M$ with involution $I$, $x\alpha I(x) = 0$ implies $x = 0, x \in M, \alpha \in \Gamma$.

**Proof.** Let $e$ be the right annihilating projection of $x$. Then $xae = 0$. Now $I(xae) = I(0) = 0$. This implies that $I(e)\alpha I(x) = 0$. So $e\alpha I(x) = 0$. Since $x\alpha I(x) = 0$, we have $I(x) \in e\Gamma M, I(x) = e\alpha I(x) = 0$. Now $x = I^2(x) = I(I(x)) = I(0) = 0$ It follows that a Baer $\Gamma$-ring with involution $I$ has no nil left or right ideals. For let $A$ be a nil right ideal in a Baer $\Gamma$-ring with involution $I$. If $x \in A$, then $y = x\alpha I(x) \in A$. If
\((ya)^ny\) is the smallest power of \(y\) that is 0, let \(z = (ya)^{n-1}y\). Then \(z\alpha I(z) = z\alpha z = 0\) whence \(z = 0\) by theorem 3.12. Hence \(x = 0\). The argument for a nil left ideal is analogous. □

A fortiori, a Baer \(\Gamma\)-ring with involution \(I\) has no nilpotent ideals. We turn now to the consideration of equivalence of projection in a Baer \(\Gamma\)-ring with involution \(I\).

**Theorem 3.6.** Let \(M\) be a Baer \(\Gamma\)-ring with involution \(I\), an element of \(M\) such that \(x\alpha I(x)\) is a projection of \(e\) for \(\alpha \in \Gamma\). Then \(I(x)\alpha x\) is also a projection of \(f\). We have \(x \in e\Gamma Me \Gamma f, I(x) \in f\Gamma Me\Gamma f\) and thus \(e \sim f\).

**Proof.** Set \(y = e\alpha x - x\). Then

\[
y\alpha I(y) = (e\alpha x - x)\alpha I(e\alpha x - x)
\]

\[
= ((e - 1)\alpha x)\alpha I(e\alpha x - x)
\]

\[
= ((e - 1)\alpha x)\alpha (I(x)\alpha I(e) - I(x))
\]

\[
= ((e - 1)\alpha x)\alpha (I(x)\alpha e - I(x))
\]

\[
= (e - 1)\alpha x\alpha I(x)\alpha e - 1)
\]

\[
= (e - 1)\alpha e\alpha (e - 1)
\]

\[
= (e\alpha e - 1)e\alpha (e - 1)
\]

\[
= (e - e)\alpha (e - 1)
\]

\[
= 0\alpha (e - 1)
\]

\[
= 0\alpha (e - 1)
\]

\[
= 0
\]

By Theorem 3.12, \(y = 0\). So \(e\alpha x - x = 0\). Thus \(e\alpha x = x\). If \(f = I(x)\alpha x\) then \(I(f) = I(I(x)\alpha x) = I(x)\alpha I^2(x) = I(x)\alpha x = f\) and \(f\alpha f = I(x)\alpha x I(x)\alpha x = \)
$I(x)aeax = I(x)ax = f$. Now $f\alpha I(x) = I(x)ax\alpha I(x) = I(x)ae = I(x)$. Thus
\[x\alpha I(x)\alpha x = e\alpha x = x, I(x) \in f\Gamma M\Gamma e.\] Hence \(e \sim f\).

**Definition 3.6.** An element \(x\) in a \(\Gamma\)-ring \(M\) with involution \(I\) is called a partial isometry if \(x\alpha I(x)\) and \(I(x)\alpha x, \alpha \in \Gamma\) are projections.

**Definition 3.7.** In a \(\Gamma\)-ring \(M\) with involution \(I\), projections \(e, f\) are called \(I\)-equivalent, written \(e \sim^I f\), if there exists a partial isometry \(x \in e\Gamma M\Gamma f\) with \(x\alpha I(x) = e, I(x)\alpha x = f\).

It is easy verified that \(\sim^I\) is an equivalence relation and that \(e \sim^I f\) implies \(e \sim f\). Note that if \(M\) is a Baer \(\Gamma\)-ring with involution \(I\) the condition \(x \in e\Gamma M\Gamma f\) in the definition of \(I\)-equivalence is redundant (Theorem 3.13). We now wish to make some comparisons between Baer \(\Gamma\)-rings and Baer \(\Gamma\)-rings with involution \(I\). We begin by exhibiting a condition that can convert a Baer \(\Gamma\)-ring into a Baer \(\Gamma\)-ring with involution \(I\).

**Theorem 3.7.** Let \(M\) be a \(\Gamma\)-ring with involution \(I\) and suppose that for every \(x\) in \(M\), \(1 + I(x)\alpha x, \alpha \in \Gamma\) is invertible in \(M\). Then for any idempotent \(f\) in \(M\) there exists a projection \(e\) such that \(f\Gamma M = e\Gamma M\).

**Proof.** Let \(x = I(f) - f\). Then \(I(x) = I(I(f) - f) = f - I(f)\). Therefore \(I(x)\alpha x = (f - I(f))\alpha (I(f) - f)\). So \(1 + I(x)\alpha x = 1 + (f - I(f))\alpha (I(f) - f)\). Since \(1 + I(x)\alpha x\) is invertible in \(M\), \(1 + (f - f - I(f))\alpha (I(f) - (f)\) is invertible in \(M\). Take \(z = 1 + (f - 1(f))\alpha (I(f) - f)\). Then, \(z\) is invertible, say \(t = z^{-1}\). Also we have, \(z = I(z)\),
then $t = I(t)$. Therefore
\[
fa z = fa(1 - f - I(f) + fαI(f) + I(f)αf) \\
= fa1 - fαf - faI(f) + fαfaI(f) + faI(f)αf \\
= f - f - faI(f) + faI(f) + faI(f)αf \\
= faI(f)αf
\]
Similarly $zαf = faI(f)αf$. It follows that $t$ commutes with $f$. We have also seen that $t$ commutes with $I(f)$. Now we choose $e = faI(f)αt$. Then $I(e) = I(fαI(f)αt) = I(t)αI^2(f)αI(f) = tαfαI(f) = fαtαI(f) = faI(f)αt = e$. Also
\[
eae = faI(f)αtαfαI(f)αt \\
= tαfαI(f)αtαfαI(f)αt \\
= tαfαI(f)αt \\
= (tαz)α(fαI(f)αt) \\
= 1ae = e
\]
Thus $e$ is a projection. Evidently $fae = e$ whence $eΓM ⊂ fΓM$. Again
\[
eaf = faI(f)αfat \\
= fazat \\
= fa(zαt) \\
= fa1 = f
\]
Therefore $fΓM ⊂ eΓM$. Hence $fΓM = eΓM$. □

**Corollary 3.2.** Let $M$ be a Baer $Γ$-ring with an involution $I$ and suppose that $1 + I(x)αx, α \in Γ$ is invertible for every $x$ in $M$. Then $M$ is a Baer $Γ$-ring with involution $I$. 
Next we give a condition which identifies the two versions of equivalence.

**Theorem 3.8.** Let $M$ be a $\Gamma$-ring with involution $I$. Assume that for any $y \in M$ there exists a self-adjoint $z \in M$ which commutes with everything that commutes with $I(y)\alpha y$ and satisfies $z \alpha z = I(y)\alpha y$, $\alpha \in \Gamma$. Then equivalent projections in $M$ are $I$-equivalent.

**Proof.** Let the projections $e, f$ be equivalent via $x, y, x \in e \Gamma M \Gamma f, y \in f \Gamma M \Gamma f, x \alpha y = e, y \alpha x = f$. Choose $z$ (relative to $y$) are permitted by the hypothesis. We have $x \alpha I(x)\alpha I(y)\alpha y = x \alpha I(y \alpha x)\alpha y = x \alpha y = e$. Since $e$ is self-adjoint, $e = I(e)$. Then $e = I(e) = I(x \alpha I(x)\alpha I(y)\alpha y) = I(y)\alpha I^2(y)\alpha I^2(x)\alpha I(x) = I(y)\alpha y \alpha x \alpha I(x)$ Therefore $x \alpha I(x)\alpha I(y)\alpha y = I(y)\alpha y \alpha x \alpha I(x)$. Thus $x \alpha I(x)$ commutes with $y \alpha I(y)$ and hence also with $z$. Now we have $I(y)\alpha y \alpha e = I(y)\alpha y$. Then $I(I(y)\alpha y \alpha e) = I(I(y)\alpha y)$. So, $I(e)\alpha I(y)\alpha I^2(y) = I(y)\alpha I^2(y)$. Thus $e \alpha I(y)\alpha y = I(y)\alpha(y)$. Hence $e \alpha I(y)\alpha y = I(y)\alpha(y)\alpha e$. Thus $eaz = zae$. The element $w = eazax \in e \Gamma M \Gamma f$ implements the desired $I$-equivalence of $e$ and $f$. Now

$$
I(w)\alpha w = I(eazax)\alpha(eazax)
$$

$$
= I(x)\alpha I(z)\alpha I(e)\alpha eazax
$$

$$
= I(x)\alpha zaeaeazax
$$

$$
= I(x)\alpha zaeaeazax
$$

$$
= I(x)\alpha zaeazax
$$

$$
= I(x)\alpha zaeazax
$$

$$
= I(x)\alpha zaeazax
$$

$$
= I(x)\alpha I(y)\alpha yax
$$

$$
= I(yax)\alpha(yax)
$$

$$
= I(f)\alpha f = f.
$$
Again we have

\[ wαI(w) = eαzαxαI(eαzαx) \]
\[ = eαzαxαI(x)αI(z)αI(e) \]
\[ = eαzαxαI(x)azαe \]
\[ = eα(xαI(x))azαe \]
\[ = eαxαI(x)azαe \]
\[ = xαI(x)αI(y)azyαe \]
\[ = xαI(yαx)azyαe \]
\[ = xαI(f)azyαe \]
\[ = xαfazyαe \]
\[ = xαfy = e. \]

Hence \( e \) and \( f \) are \( I \)-equivalent.

□

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