On Homogeneous Bitopological Spaces

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Abstract

We shall introduce the homogeneity concept in bitopological spaces. We shall also define homogeneity components in these spaces. We shall discuss the relation between homogeneity components in both single topologies, homogeneity components in the bitopological spaces and the homogeneity components in the least upper bound topology that is generated by both single topologies. The relation between the homogeneity of the bitopological spaces, the homogeneity of the single topologies and the homogeneity of the least upper bound topology will be also discussed. Relevant examples will be illustrated in the paper.

Introduction

In 1963, Kelly [2] introduced the notion of a bitopological space which is the triple \((X, \tau_1, \tau_2)\), where \(X\) is a nonempty set and \(\tau_1\) and \(\tau_2\) are two topologies on \(X\). Later, several authors had studied this notion and other related concepts. In particular, Fora and Al-Refaei (see [1]) discussed the fixed point theory in bitopological spaces. They have the following.

Definition 1.1.

Let \(\tau_1\) and \(\tau_2\) be two topologies on \(X\). Then \(\tau_1 \cup \tau_2\) forms a subbase for some topology on \(X\). This topology is called the least upper bound topology on \(X\), and is denoted by \(<\tau_1, \tau_2>\).

In [1], Fora and Al-Refaei had clarified the relation between \((X, \tau_1), (X, \tau_2)\) and \((X, <\tau_1, \tau_2>)\). They had the following result.
Theorem 1.2. Let \( (X, \tau_1, \tau_2) \) be a bitopological space and let \( \Delta = \{(x,x) : x \in X\} \) be the diagonal subspace of the product of the two topological spaces \( (X, \tau_1) \) and \( (X, \tau_2) \). Then \( (X, <\tau_1, \tau_2>) \) is isomorphic to \( \Delta \).

Let \( (X, \tau) \) be a topological space. By \( H(X, \tau) \) (we shall abbreviate it by \( H(X) \)) if there will be no confusion arise we mean the collection of all homeomorphisms from \( (X, \tau) \) onto itself. If \( \rho \) is a relation defined on \( X \) as follows: \( (x,y) \in \rho \) if and only if there is an \( h \in H(X) \) such that \( h(x) = y \). Then \( \rho \) is an equivalence relation on \( X \). The equivalence class of \( X \) that contains \( x \) will be denoted by \( C_x \) and is called the homogeneous component of \( (X, \tau) \) determined by \( x \). In the case of a bitopological space \( (X, \tau_1, \tau_2), C_x^1 \) and \( C_x^2 \) will be the homogeneous components of \( (X, \tau_1) \) and \( (X, \tau_2) \), respectively, that determined by \( x \). However, \( C_x^{12} \) will denote the homogeneous component of \( (X, <\tau_1, \tau_2>) \) that contains \( x \).

A topological space \( (X, \tau) \) is called homogeneous provided it has only one homogeneous component (namely, \( X \) itself). A topological space \( (X, \tau) \) is called rigid if \( H(X) \) is a single point set (i.e., the identity mapping is the only homeomorphism from \( (X, \tau) \) onto itself). It is easy to see that a topological space \( (X, \tau) \) is rigid if and only if \( C_x = \{x\} \) for each \( x \in X \).

\( \mathbb{N} \) and \( \mathbb{R} \) will denote the sets of all natural and real numbers, respectively. \( \tau_{l,r} \), \( \tau_{r,r} \) and \( \tau_u \) will denote the left ray, the right ray and the usual (Euclidean) topologies on \( \mathbb{R} \), respectively. If \( X \) is a set then \( \tau_{dis} \) will denote the discrete topology on \( X \).

2. Homogeneity components in bitopological spaces. We start this section with the following definition.

Definition 2.1. Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \tau'_1, \tau'_2) \) be a mapping between two bitopological spaces \( (X, \tau_1, \tau_2) \) and \( (Y, \tau'_1, \tau'_2) \).

(i) If \( f : (X, \tau_1) \rightarrow (Y, \tau'_1) \) and \( f : (X, \tau_2) \rightarrow (Y, \tau'_2) \) are both continuous mappings, then \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \tau'_1, \tau'_2) \) is called a continuous mapping.

(ii) If \( f : (X, \tau_1) \rightarrow (Y, \tau'_1) \) and \( f : (X, \tau_2) \rightarrow (Y, \tau'_2) \) are both open mappings, then \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \tau'_1, \tau'_2) \) is called an open mapping.

(iii) If \( f : (X, \tau_1) \rightarrow (Y, \tau'_1) \) and \( f : (X, \tau_2) \rightarrow (Y, \tau'_2) \) are both homeomorphisms, then \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \tau'_1, \tau'_2) \) is called a homeomorphism.
Although $H_{12}(X) = H(X, \tau_1) \cap H(X, \tau_2)$ but this does not reflect the same relation on homogeneous components as we shall see later.

Now, we can give our first result in this section that generalizes the well known homogeneity components in single topologies.

**Theorem 2.2.** Let $(X, \tau_1, \tau_2)$ be a bitopological space. Define a relation $\rho$ on $X$ as follows: $(x,y) \in \rho$ if and only if there is an $h \in H_{12}(X)$ such that $h(x) = y$. Then $\rho$ is an equivalence relation on $X$.

**Proof**

To prove the reflexivity of $\rho$, notice that the identity mapping $i: (X, \tau_1, \tau_2) \rightarrow (X, \tau_1, \tau_2)$, defined by $i(x) = x$, is a homeomorphism that sends $x$ to itself. For the symmetry of $\rho$, notice that if $f \in H_{12}(X)$ such that $f(x) = y$ then $f^{-1} \in H_{12}(X)$ satisfies the condition that $f^{-1}(y) = x$. For the transitivity of $\rho$, notice that if $f, g \in H_{12}(X)$ such that $f(x) = y$ and $g(y) = z$, then $g \circ f \in H_{12}(X)$ satisfies the condition that $(g \circ f)(x) = z$.

**Definition 2.3.** The equivalence class of $X$ induced by the equivalence relation $\rho$; that was defined in Theorem 2.2; and contains $x$ will be denoted by $C_x^b$ and is called the (bitopological) homogeneous component of $X$ determined by $x$.

The following result clarifies the relation between the homogeneous components $C_x^1$, $C_x^2$ and $C_x^b$.

**Theorem 2.4.** Let $(X, \tau_1, \tau_2)$ be a bitopological space and let $x \in X$. Then we have the following:

i) $C_x^b \subseteq C_x^1 \cap C_x^2$.

ii) The equality in (i) is in general false.

**Proof**

i) Let $y \in C_x^b$. Then there is an $h \in H_{12}(X)$ such that $h(x) = y$. It follows that $h \in H(X, \tau_1)$ and $h \in H(X, \tau_2)$. Hence $y \in C_x^1$ and $y \in C_x^2$, i.e. $y \in C_x^1 \cap C_x^2$.

ii) To provide an example, let $X = \{1, 2, 3, 4, 5, 6\}$, $\beta_1 = \{\{1, 2, 5\}, \{3, 4, 6\}\}$ and $\beta_2 = \{\{1, 3, 5\}, \{2, 4, 6\}\}$. Then $\beta_1$ and $\beta_2$ are bases for the topologies $\tau_1 = \tau(\beta_1)$ and $\tau_2 = \tau(\beta_2)$ on $X$. It is clear that both topological spaces $(X, \tau_1)$ and $(X, \tau_2)$ are homogeneous. Hence $C_x^1 \cap C_x^2 = X$ for every $x \in X$. We claim that there is no $h \in H_{12}(X)$ such that $h(1) = 2$. To prove our claim, suppose on the contrary, i.e. there is an $h \in H_{12}(X)$ such that $h(1) = 2$. Since $h \in H(X, \tau_1)$ therefore $h(\{1, 2, 5\}) = \{1, 2, 5\}$. Hence $h(3) \in \{1, 5\}$ because $h(1) = 2$. Similarly, since $h \in H(X, \tau_2)$ therefore $h(\{1, 3, 5\}) = \{2, 4, 6\}$. Hence $h(5) \in \{4, 6\}$.
because \( h(1)=2 \). This, of course, will yield to a contradiction because \( h(5) \in \{1,5\} \cap \{4,6\} = \emptyset \). This completes the proof of our claim. Consequently \( 2 \notin C^b_1 \), i.e. \( C^b_1 \neq X \).

To state our next result, we need the following lemma.

**Lemma 2.5.** Let \( f : X \to X \) be a bijection and let \( (X, \tau_1, \tau_2) \) be a bitopological space.

i) \( ff : (X, \tau_1, \tau_2) \to (X, \tau_1, \tau_2) \) is a continuous mapping, then the mapping
\( f : (X, <\tau_1, \tau_2>) \to (X, <\tau_1, \tau_2>) \) is also continuous.

ii) \( ff : (X, \tau_1, \tau_2) \to (X, \tau_1, \tau_2) \) is an open mapping, then the mapping
\( f : (X, <\tau_1, \tau_2>) \to (X, <\tau_1, \tau_2>) \) is also open.

iii) \( ff : (X, \tau_1, \tau_2) \to (X, \tau_1, \tau_2) \) is a homeomorphism, then the mapping
\( f : (X, <\tau_1, \tau_2>) \to (X, <\tau_1, \tau_2>) \) is also a homeomorphism.

**Proof.**

i) Let \( V \) be a basic open set in \( (X, <\tau_1, \tau_2>) \). Then there are sets \( V_i \in \tau_i \) \( (i=1,2) \) in \( X \) such that \( V = V_1 \cap V_2 \). Since \( f : (X, \tau_1) \to (X, \tau_1) \) and \( f : (X, \tau_2) \to (X, \tau_2) \) are continuous mappings, therefore \( f^{-1}(V_i) \in \tau_i \) for \( i=1,2 \). Consequently \( f^{-1}(V) = f^{-1}(V_1) \cap f^{-1}(V_2) \in <\tau_1, \tau_2> \), i.e. \( f^{-1}(V) \) is open in \( (X, <\tau_1, \tau_2>) \), and this completes the proof.

ii) Its proof is similar to (i) by noticing that \( f \) is an injective mapping.

iii) Its proof is a direct consequence of (i) and (ii).

It is important to notice that the converse of Lemma 2.5 is indeed false. To see this, we propose the following example.

**Example 2.6.** Let \( X = \mathbb{R} \), \( \tau_1 = \tau_{\ell, r} \) and \( \tau_2 = \tau_{r, r} \). Let \( f : \mathbb{R} \to \mathbb{R} \) be the mapping defined by \( f(x) = -x \). Then \( <\tau_1, \tau_2> = \tau_u \) and \( f \in H(X, <\tau_1, \tau_2>) \) but \( f : (X, \tau_1) \to (X, \tau_1) \) is neither continuous nor open. Notice also that \( f : (X, \tau_2) \to (X, \tau_2) \) is neither continuous nor open.

Let us state our next result concerning the homogeneity components in bitopological spaces.

**Theorem 2.7.** Let \( (X, \tau_1, \tau_2) \) be a bitopological space and let \( x \in X \). Then we have the following:

i) \( C^b_x \subseteq C^b_x^{12} \).
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ii) The equality in (i) is in general false.

Proof

i) Let $y \in C_x^b$. Then there is an $h \in H_{12}(X)$ such that $h(x) = y$. Using Lemma 2.5, we have $h \in H(X, \tau_1, \tau_2)$ satisfies the condition that $h(x) = y$. Hence $y \in C_x^{12}$.

ii) To provide an example, let $X = \{1,2,3\}$, $\tau_1 = \{\phi, X, \{1\}, \{1,2\}\}$ and $\tau_2 = \{\phi, X, \{3\}, \{3,2\}\}$. Then $<\tau_1, \tau_2> = \tau_{ab}$ and consequently, we have $C_x^{12} = X$ for all $x \in X$. Since $(X, \tau_1)$ is a rigid space (see Remark 3.2), therefore $C_x^b = \{x\}$ for every $x \in X$.

Theorems 2.4 and 2.7 state that homogeneity components in bitopological spaces are weaker (contained in) than homogeneity components in both single topologies and also weaker than homogeneity components in the least upper bound topology.

To make this section complete, we present the following result.

Theorem 2.8. Let $(X, \tau_1, \tau_2)$ be a bitopological space and let $x \in X$. Then we have the following:

i) The inclusion $C_x^1 \cap C_x^2 \subseteq C_x^{12}$ is in general false.

ii) The inclusion $C_x^{12} \subseteq C_x^1 \cap C_x^2$ is in general false.

Proof:

i) To provide an example, let $X = \{1,2,3,4,5,6\}$, $\beta_1 = \{\{1,2,5\}, \{3,4,6\}\}$ and $\beta_2 = \{\{1,3,5\}, \{2,4,6\}\}$. Then $\beta_1$ and $\beta_2$ are bases for the topologies $\tau_1 = \tau(\beta_1)$ and $\tau_2 = \tau(\beta_2)$ on $X$ (see Theorem 2.4 (ii)). Notice that $\{2\} \subseteq <\tau_1, \tau_2>$ and $\{x\} \not\subseteq <\tau_1, \tau_2>$ for all $x \in X \setminus \{2,3\}$. This implies that $C_x^{12} = \{2,3\}$. Hence $C_x^1 \cap C_x^2 = X$, which is obviously not contained in $C_x^{12}$.

ii) To provide an example, let $X = \{1,2\}$, $\tau_1 = \{\phi, X, \{1\}\}$ and $\tau_2 = \{\phi, X, \{2\}\}$. Then $<\tau_1, \tau_2> = \tau_{ab}$ and hence $C_x^{12} = X$ for all $x \in X$. Since $(X, \tau_i)$ are rigid spaces ($i = 1,2$), therefore $C_x^i = \{x\}$ for all $x \in X, i = 1,2$. Therefore $C_x^1 \cap C_x^2 = \{x\}$. Now, it is clear that $C_x^{12} = X$ is not contained in $C_x^1 \cap C_x^2 = \{x\}, x \in X$.

3. Homogeneous bitopological spaces. We start this section with the following definition that generalizes the homogeneity and the rigidity concepts in single topologies (Of course, in the case that the two topologies are equal).

Definition 3.1. A bitopological space $(X, \tau_1, \tau_2)$ is called:

i) homogeneous if $C_x^b = X$ for all (equivalently, for some) $x \in X$. 

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ii) rigid provided that $H_{12}(X)$ is a single point set.

We need the following remark for single topologies.

**Remark 3.2.** If $X = \{ 1, 2, \ldots, n \}$ has the topology $\tau = \{ \emptyset, \{ 1, 2, \ldots, k \} : k = 1, 2, \ldots, n \}$ (i.e. $\tau = \{ \emptyset, \{ 1, \}, \{ 1, 2 \}, \{ 1, 2, 3 \}, \ldots, \{ 1, 2, \ldots, n \} \}$). Then $(X, \tau)$ is a rigid space.

**Proof**

Let $h \in H(X)$. We are going to prove that $h(k) = k$ for each $k = 1, 2, \ldots, n$. The proof is by induction. Since $h : (X, \tau) \to (X, \tau)$ is an open mapping and $\{ 1 \} \in \tau$, therefore $\{ h(1) \} \in \tau$. Hence $h(1) = 1$. Now, assume that $h(k) = k$ is true for all $1 \leq k < m \leq n$. Since $\{ 1, 2, \ldots, m-1, m \} \in \tau$ and $h$ is an open mapping, therefore $h(\{ 1, 2, \ldots, m-1, m \}) \in \tau$. Since $h$ is an injective mapping, therefore $h(m) = m$ and this completes our proof.

**Theorem 3.3.** Let $(X, \tau_1, \tau_2)$ be a bitopological space.

i) If $(X, \tau_1)$ or $(X, \tau_2)$ is a rigid space, then $(X, \tau_1, \tau_2)$ is a rigid space (and hence $C^h_x = \{ x \}$ for all $x \in X$).

ii) The rigidity of $(X, \tau_1, \tau_2)$ does not imply the rigidity of $(X, \tau_1)$ or the rigidity of $(X, \tau_2)$.

**Proof**

i) Notice that $H_{12}(X) = H(X, \tau_1) \cap H(X, \tau_2)$. Thus, if $H(X, \tau_1)$ or $H(X, \tau_2)$ is a single point set, then $H_{12}(X)$ will also be a single point set, i.e. $H_{12}(X)$ is consisting of the identity mapping only. Hence $(X, \tau_1, \tau_2)$ is a rigid space.

ii) To provide an example, let $X = \{ 1, 2, 3 \}$, $\tau_1 = \{ \emptyset, X, \{ 1, 2 \}, \{ 3 \} \}$ and $\tau_2 = \{ \emptyset, X, \{ 1, 3 \}, \{ 2 \} \}$. If $f \in H_{12}(X)$ then $f \in H(X, \tau_1)$ and $f \in H(X, \tau_2)$. Since $\{ 3 \} \in \tau_1$, therefore $f(\{ 3 \}) = f(\{ 3 \}) \in \tau_1$. Thus $f(3) = 3$. Similarly, since $\{ 2 \} \in \tau_2$ therefore $f(\{ 2 \}) = f(\{ 2 \}) \in \tau_2$ and hence $f(2) = 2$. But $f$ is a bijection, hence it follows that $f(1) = 1$. Consequently, $f$ is the identity mapping and thus $(X, \tau_1, \tau_2)$ is a rigid space. Notice that neither $(X, \tau_1)$ nor $(X, \tau_2)$ are rigid spaces because $C^1_x = \{ 1, 2 \}$ and $C^2_x = \{ 1, 3 \}$.

**Example 3.4.** There exists a bitopological space $(X, \tau_1, \tau_2)$ such that $(X, \tau_1)$ and $(X, \tau_2)$ are rigid spaces but $(X, \tau_1, \tau_2)$ is a homogeneous space.
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Proof

Take $X = \{1,2\}, \tau_1 = \{\emptyset, X, \{1\}\}$ and $\tau_2 = \{\emptyset, X, \{2\}\}$. Then all of the above mentioned properties are satisfied because of Remark 3.2 and the fact that $<\tau_1, \tau_2> = \tau_{dis}$. 

Theorem 3.5. Let $(X, \tau_1, \tau_2)$ be a bitopological space.

i) If $(X, \tau_1, \tau_2)$ is homogeneous, then $(X, \tau_1)$ and $(X, \tau_2)$ are homogeneous spaces.

ii) The converse of (i) is in general false.

Proof

(i) Its proof is straightforward.

(ii) The example in Theorem 2.4(ii) will satisfy our need.

Theorem 3.6. Let $(X, \tau_1, \tau_2)$ be a bitopological space.

i) If $(X, \tau_1, \tau_2)$ is homogeneous, then $(X, <\tau_1, \tau_2>)$ is homogeneous.

ii) The converse of (i) is in general false.

Proof

i) Since $(X, \tau_1, \tau_2)$ is homogeneous, therefore $C^b_x = X$ for each $x \in X$. Using Theorem 2.7, we get $X \subseteq C^i_x$ for all $x \in X$. Therefore $C^i_x = X$ and consequently $(X, <\tau_1, \tau_2>)$ is a homogeneous space.

ii) The example in Theorem 3.3(ii) will satisfy our need because $<\tau_1, \tau_2> = \tau_{dis}$.

Finally, we have the following result.

Theorem 3.7. Let $(X, \tau_1, \tau_2)$ be a bitopological space. Then we have the following:

i) The homogeneity of the spaces $(X, \tau_1)$ and $(X, \tau_2)$ does not imply the homogeneity of $(X, <\tau_1, \tau_2>)$.

ii) The homogeneity of $(X, <\tau_1, \tau_2>)$ does not imply the homogeneity of $(X, \tau_1)$ or the homogeneity of $(X, \tau_2)$.

Proof

i) To provide an example, consider $X = \{1,2,3,4,5,6\}$, $\beta_1 = \{\{1,2,5\}, \{3,4,6\}\}$ and $\beta_2 = \{\{1,3,5\}, \{2,4,6\}\}$. Then $\beta_1$ and $\beta_2$ are bases for the topologies $\tau_1 = \tau(\beta_1)$ and $\tau_2 = \tau(\beta_2)$ on $X$ (see Theorem 2.4(ii) and Theorem 2.8(i)). It is clear that the spaces $(X, \tau_1)$ and $(X, \tau_2)$ are homogeneous. However, as

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we have explained before; the space \((X,<\tau_1,\tau_2>)\) is not homogeneous because \(C_{12} = \{2,3\} \neq X\).

iii) To provide an example, consider \(X = \{1,2,3\}, \tau_1 = \{\emptyset, X, \{1,2\}, \{3\}\} \) and \(\tau_2 = \{\emptyset, X, \{1,3\}, \{2\}\}\). Then \(<\tau_1,\tau_2> = \tau_{div}\), and hence \((X,<\tau_1,\tau_2>)\) is a homogeneous space. Notice that neither \((X,\tau_1)\) nor \((X,\tau_2)\) are homogeneous spaces.

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References


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