COMPACTIFICATIONS AND F-SPECTRAL SPACES

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Abstract. If $X$ is $T_3$, it is showed that the Fan-Gottesman compactification of $X$ can be embedded into compactification $(X^*, k)$ of $X$ obtaining by a combined approach of nets and open filters. By F-spectral, we mean a topological space $X$ such that the Fan-Gottesman compactification of $X$ is a spectral space. We give necessary and sufficient conditions on $X$ in order to get F-spectral.

1. Introduction

The first section of this paper contains some preliminaries about net, filters and a process of obtaining a compactification $(X^*, k)$ of an arbitrary topological space $X$. In 2005, Hueytzen J. Wu and Wan-Hong Wu described a process of obtaining a compactification of an arbitrary topological space by a combined approach of nets and open filters. Besides they showed the relation among Wallman, Stone-Cech and $(X^*, k)$ compactification under some conditions [12].

In the second section of our paper contains some information about Wallman and Fan-Gottesman compactification. In 1938, Henry Wallman introduced compactification of $T_1$ spaces having a normal base [6],[9] which is also called Wallman compactification [10]. In 1952, Ky Fan ve Noel Gottesman constructed a compactification, also called Fan-Gottesman compactification, for a regular space with a normal base

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Their method is similar to Wallman compactification. In [5] it is investigated relation between Fan-Gottesman and Wallman compactifications and showed that Fan-Gottesman compactification of some interesting and specific spaces such as normal $A_2$ and $T_4$ is Wallman-type compactification. At the end of this section, if studied space is $T_3$, we show that the Fan-Gottesman compactification of $X$ can be embedded into the compactification $(X^*, k)$. Also we examined the relation between Wallman and Fan-Gottesman compactification via net and filters.

In the third section of this paper contains some preliminaries about $T_0$ compactification and spectral spaces. In 1993 Herrlich has constructed [7] with any $T_0$-space $X$, a minimal compactification $\beta_{w}X$ called the $T_0$-compactification of $X$. For $T_1$ space, the extension $\beta_{w}X$ coincides with the Wallman compactification $\gamma X$ of $X$. In 2004 Karim Belaid, Othman Echi and Riyadh Gargouri [1] have characterized topological spaces $X$ such that one point compactification of $X$ is a spectral space. In 2006 Karim Belaid [2], gave some properties of H-spectral space which he means a topological space $X$ such that its $T_0$-compactification is spectral. Also he gave necessary and sufficient condition on the $T_1$-space $X$ in order to get its Wallman compactification spectral. At the end of this section, we define F-spectral spaces and investigate necessary and sufficient condition in order that Fan-Gottesman compactification of $T_3$-space is spectral.

2. Nets, filters and $(X^*, k)$ compactification

Let $A$ be a family of continuous functions on a topological space $X$. A net $(x_\lambda)$ in $X$ will be called an $A-$net, if $(f(x_\lambda))$ converges for each $f$ in $A$. Then $X$ is compact if

1. $f(X)$ is contained in a compact subset $C_f$ for each $f$ in $A$, and
2. Every $A-$net has a cluster point in $X$. 

Let $X$ be any arbitrary topological space, $C^*(X) = \{f_\alpha : \alpha \in \Lambda\}$ the family of all bounded real-valued continuous functions on $X$. For a $C^*(X)$-net $(x_i)$, let

$$\mathcal{F}_{(x_i)} = \{ U : U \text{ is open in } X \text{ and } (x_i) \text{ residually in } U \}$$

It is clear that $\mathcal{F}_{(x_i)}$ is an open filter, and for any $f_\alpha \in C^*(X)$, any $\varepsilon > 0$, $f_\alpha^{-1}((\delta_\alpha - \varepsilon, \delta_\alpha + \varepsilon)) \in \mathcal{F}_{(x_i)}$, where $\delta_\alpha = \lim (f_\alpha(x_i))$. It is called $\mathcal{F}_{(x_i)}$ the open filter on $X$ induced by $(x_i)$.

**Definition 2.1.** If $F$ is a filter on $X$, let $\Lambda_F = \{(x, F) : x \in F \subset F\}$. Then $\Lambda_F$ is directed by the relation $(x_1, F_1) \leq (x_2, F_2)$ if $F_2 \subset F_1$, so the map $P : \Lambda_F \to X$ defined by $P(x, F) = x$ is a net in $X$. It is called the net based on $F$.

**Lemma 2.1.** A filter $F$ converges to $x$ in $X$ if the net based on $F$ converges to $x$.

**Corollary 2.1.** Let $Q$ be an open filter on $X$, $(x_i)$ is the net based on $Q$, and

$$I = \{ U : U \text{ is open in } X \text{ and } (x_i) \text{ is in } U \}$$

Then $I = Q$.

For each $C^*(X)$-net $(x_i)$ in $X$, let $\left( w_n^{(x_i)} \right)$ be the net based on the open filter $\mathcal{F}_{(x_i)}$ induced by $(x_i)$. It is clear by Definition 2.1, Lemma 2.1, and corollary 2.1. that:

1. $\left( w_n^{(x_i)} \right)$ is uniquely determined by $\mathcal{F}_{(x_i)}$ and $\mathcal{F}_{(x_i)} = \mathcal{F}_{(x_j)}$ if $\left( w_n^{(x_i)} \right) = \left( w_n^{(x_j)} \right)$

2. $\mathcal{F}_{(x_i)} = \mathcal{F}_{w_n^{(x_i)}} = \{ G : G \text{ is open in } X \text{ and } \left( w_n^{(x_i)} \right) \text{ is residually in } G \}$

3. $\left( w_n^{(x_i)} \right)$ is a $C^*(X)$-net and $\lim (f_\alpha \left( w_n^{(x_i)} \right)) = \lim (f_\alpha(x_i))$ for all $f_\alpha$ in $C^*(X)$

4. The following are equivalent:
   a. $\left( w_n^{(x_i)} \right)$ converges to $x$,
   b. $(x_i)$ converges to $x$
   c. $\mathcal{F}_{(x_i)}$ converges to $x$. 
In order to avoid the confusion between \( (w_n(x_i))^* \) as a net in \( X \) and \( (w_n(x_i)) \) as a point in a set, we will use \( (w_n(x_i))^* \) to represent \( (w_n(x_i)) \) when it regarded as a point in a set just as in [12] .

Let \( Y = \{ (w_n(x_i))^* : (x_i) \text{ is a } \text{C}^* \text{ - net that does not converge in } X \} \) and it is noted that \( (w_n(x_i)) \) is the net based on \( F_{(x_i)} \). \( X^* = X \cup Y \), the disjoint union of \( X \) and \( Y \). For each open set \( U \subset X \), define \( U^* \subset X^* \) to be the set

\[
U^* = U \cup \left\{ (w_n(x_i))^* : (w_n(x_i))^* \in Y \text{ and } (w_n(x_i)) \text{ is residually in } U \right\}
\]

It is clear that if \( U \subset V \), then \( U^* \subset V^* \). It is seen that \( \beta = \{ U^* : U \text{ is open in } X \} \) is a base for a topology on \( X^* \).

Let \( k : X \rightarrow X^* \) be defined by \( k(x) = x \). Then \( k \) is a continuous function from \( X \) into \( X^* \). Moreover \( k(X) \) is dense in \( X^* \) and \( (X^*, k) \) is compactification of \( X \).

Let us cite [11],[12] for detailed information about this section.

3. Wallman and Fan Gottesman compactification

The Wallman compactification is defined in [11] as follows.

Let \( X \) be a \( T_2 \) space and \( \gamma X \) be the collection of all closed ultrafilters on \( X \). For each closed set \( D \subset X \), define \( D \subset \gamma X \) to be the set \( D = \{ F \in \gamma X : D \in F \} \). Let \( \zeta = \{ D : D \text{ is closed subset of } X \} \) be the base for the closed sets of the topology on \( \gamma X \), and let \( h : X \rightarrow \gamma X \) be defined by \( h(x) = F_x \), the closed ultrafilter converging to \( x \) in \( X \). Then \( (\gamma X, h) \) is the Wallman compactification of \( X \).

Now we investigate how Wallman compactification is obtained via normal base.

Let \( \beta \) is a class of closed sets in \( X \). If it satisfies following three conditions, \( \beta \) is called normal base.

1) \( \beta \) is closed under finite intersection and unions.
2) If \( x \) is not contained in the closed set \( A \), there is a set \( B \in \beta \) such that \( x \in B \subset X - A \).
3) If \( A_1 \cap A_2 = \emptyset \), for \( \forall A_1, A_2 \in \beta \), there exist sets \( A_m, A_n \in \beta \) such that \( A_1 \subset X - A_n, A_2 \subset X - A_m, A_n \cup A_m = X \).

Let \( X \) be a \( T_1 \) space having a normal base and \( \beta \) be a normal base in \( X \). It is considered \( K \) space whose element is denoted by letter as \( a', b', \ldots \) consist of finite number of \( F_i \) in \( X \) such that

\[
F_1 \cap F_2 \cap F_3 \cap \ldots \cap F_n \neq \emptyset
\]

and maximal with respect to above property. Let \( \tau (F) = \{a' \in K : F \in a'\} \). It is defined topology of \( K \) with a family of sets \( \delta = \{\tau (F) : F \in \beta\} \) a base of closed set. \( K \) is a compact space and compactification of \( X \). This compactification is called Wallman compactification [6],[9],[10]. In order to avoid the confusion it is denoted by \( \gamma X \).

There is very little difference between Fan-Gottesman and Wallman compactification, \( \beta \) forming Wallman compactification is a normal base for closed sets but \( \beta \) forming Fan-Gottesman compactification is a normal base for open sets. It shall not be forgotten that both of these satisfy conditions of normal base.

It is considered that \( X \) is a regular space having a base for open set \( \beta \) which satisfies above three properties of normal base. But Ky Fan and Noel Gottesman used for any \( A \in \beta \) and any open set \( G \) of \( X \) such that \( cl_x A \subset G \), there exist a \( B \in \beta \) such that \( cl_x A \subset B \subset cl_x B \subset G \), where closure of \( A \) in \( X \) will be denoted \( cl_x A \), instead of if \( A_1 \cap A_2 = \emptyset \), for \( \forall A_1, A_2 \in \beta \), there exist sets \( A_m, A_n \in \beta \) such that

\[
A_1 \subset X - A_n, A_2 \subset X - A_m, A_n \cup A_m = X
\]

A chain family on \( \beta \) is a non-empty family of sets of \( \beta \) such that

\[
cl_x A_1 \cap cl_x A_2 \cap cl_x A_3 \cap \ldots \cap cl_x A_n \neq \emptyset
\]
for any finite number of sets $A_i$ of the family. Every chain family on $\beta$ is contained in at least one maximal chain family on $\beta$ by Zorn's lemma. Maximal chain families on $\beta$ will be denoted by letters as $a^*, b^*, ...$ and also the set of all maximal chain families on $\beta$ will be denoted by $FX$. $FX$ is a compact, hausdorff spaces and compactification of regular spaces $X$. This compactification is called Fan-Gottesman compactification [6].

We know the relation between Wallman and Fan-Gottesman compactifications of some specific spaces from [5]. Therefore, we can obtain the Fan-Gottesman compactification by defining the base via nets and filters like the Wallman compactification.

**Definition 3.1.** Let $X$ be a $T_3$ space and $\kappa X$ the subcollection of all open ultrafilters on $X$. For each open set $O \subset X$, define $O^* \subset \kappa X$ to be the set

$$ O^* = \left\{ \hat{G} \in \kappa X : O \subset \text{cl}_X O \subset V, \text{V is open in } X \text{ and } V \in \hat{G} \right\} $$

Let $\Phi$ is the $\{O^* : O \text{ is open subset of } X\}$ set. It is clear that $\Phi$ is the base for open sets of topology on $\kappa X$. $\kappa X$ is a compact space and the Fan-Gottesman compactifications of $X$. In order to avoid the confusion it is denoted by $\kappa X$.

On the other hand, for each closed set $D \subset X$, we define $D^* \subset \kappa X$ by

$$ D^* = \left\{ \hat{G} \in \kappa X : G \subseteq D \text{ for some } G \text{ in } \hat{G} \right\} $$

The following properties of $\kappa X$ are useful

1. If $U \subset X$ is open, then $\kappa X - U^* = (\kappa X - U)^*$
2. If $D \subset X$ is closed, then $\kappa X - D^* = (\kappa X - D)^*$
3. If $U_1$ and $U_2$ are open in $X$, then $(U_1 \cap U_2)^* = U_1^* \cap U_2^*$ and $(U_1 \cup U_2)^* = U_1^* \cup U_2^*$

**Theorem 3.1.** The Fan-Gottesman compactification $\kappa X$ of $X$ can be embedded into the compactification $(X^*, k)$ of $X$, if $X$ is $T_3$. 
Proof. We must define a map from $\kappa X$ to $(X^*, k)$ and show that the map is an embedding.

Firstly, let $(X^*, k)$ be compactification of $X$ defined as section 1.

Let $\varphi : \kappa X \to (X^*, k)$ be defined by setting that $\varphi(\hat{G}_x) = x$, if $\hat{G}_x$ is the open ultrafilters converging to $x$ in $X$; $\varphi(\hat{G}) = \left(\left(w_n(\hat{G})\right)^*\right)$ and $\left(\left(w_n(\hat{G})\right)\right)$ is the net based on $\hat{G}$, moreover $\left(\left(w_n(\hat{G})\right)\right)$ is the ultranet in $X$, if $\hat{G}$ is open ultrafilter that does not converge in $X$. That is:

$$\varphi = \begin{cases} x & \text{if } \hat{G}_x \text{ is the open ultrafilters converging to } x \text{ in } X \\ \left(\left(w_n(\hat{G})\right)\right)^* & \text{if } \hat{G} \text{ is open ultrafilter that does not converge in } X \end{cases}$$

From conclusion of Lemma 2.1., $\left(\left(w_n(\hat{G})\right)\right)$ is a $C^*(X) - net$ that does not converge in $X$. Since $\left(\left(w_n(\hat{G})\right)\right)$ is the net based on $\hat{G}$ thus by corollary 2.1., the open filter $\hat{G}_{w_n(\hat{G})}$ induced by $\left(\left(w_n(\hat{G})\right)\right)$ is exactly $\hat{G}$. Hence $\left(\left(w_n(\hat{G})\right)\right)$ is in $Y$ defined as section 2. Since $X$ is a $T_3$, $X$ is a Hausdorff then for $\forall \, x \neq y$ there exist open neighborhoods $U_x$ of $x$ and $U_y$ of $y$ such that $U_x \cap U_y = \emptyset$. $G_x$ converging to $x$ and $G_y$ converging to $y$ imply that $U_x \supset A$ for some $A \in G_x$ and $U_y \supset B$ for some $B \in G_y$.

If $G_x = G_y$ then $A$ and $B$ are both in $G_x$ and $A \cap B \neq \emptyset$. Hence $U_x \cap U_y \supset A \cap B \neq \emptyset$. This contradicts the fact that $U_x \cap U_y = \emptyset$. So $G_x = G_y$ implying $x = y$. Therefore both $\hat{G}$ and $\left(\left(w_n(\hat{G})\right)\right)$ are uniquely determined by a given open ultrafilter $\hat{G}$ that does not converge in $X$. Thus $\varphi$ is well-defined.

Secondly, we show that $\varphi$ is an injective map.

1) If $\hat{G}_x$ and $\hat{G}_y$ are two open ultra filters converging to $x$ and $y$, respectively, and $\hat{G}_x \neq \hat{G}_y$. Then $\varphi(\hat{G}_x) = x$ and $\varphi(\hat{G}_y) = y$. Then, there exist $U_0 \in \hat{G}_x$ and $V_0 \in \hat{G}_y$ such that $U_0 \cap V_0 = \emptyset$. Since $\hat{G}_x$ converges to $x$ and $\hat{G}_y$ converges to $y$, so $x \in U$ for all $U \in \hat{G}_x$ and $y \in V$ for all $V \in \hat{G}_y$. Thus $U_0 \cap V_0 = \emptyset$ implies that $x \neq y$. 


2) If $\hat{G}_1, \hat{G}_2$ are two open ultra filters that don’t converge in $X$ and $\hat{G}_1 \neq \hat{G}_2$, then $\varphi(\hat{G}_1) = \left( w_n^{(\hat{G}_1)} \right)^* \text{ and } \varphi(\hat{G}_2) = \left( w_n^{(\hat{G}_2)} \right)^*$. Since $\hat{G}_1, \hat{G}_2$ are two different open ultra filters, the nets $\left( w_n^{(\hat{G}_1)} \right)$ and $\left( w_n^{(\hat{G}_2)} \right)$ based on $\hat{G}_1$ and $\hat{G}_2$, respectively, are different. That is $\left( w_n^{(\hat{G}_1)} \right) \neq \left( w_n^{(\hat{G}_2)} \right)$. Hence $\left( w_n^{(\hat{G}_1)} \right)^* \neq \left( w_n^{(\hat{G}_2)} \right)^*$ in $Y$.

3) If $\hat{G}_x$ is an open ultra filters converging to $x$ in $X$ and $\hat{G}$ is a open ultra filters that does not converge in $X$, then $\hat{G}_x \neq \hat{G}$. Since $\varphi(\hat{G}_x) = x \in X$, $\varphi(\hat{G}) = \left( w_n^{(\hat{g})} \right)^* \in Y$ and $X \cap Y = \emptyset$, so $\varphi(\hat{G}_x) \neq \varphi(\hat{G})$. Therefore, $\varphi$ is one to one.

Thirdly, $\varphi$ and $\varphi^{-1}$ are continuous. Let $U^*$ be open set in $\beta$ defined as section 2; i.e., $U^* = U \cup \left\{ \left( w_n^{(x_i)} \right)^* : (w_n^{(x_i)})^* \in Y \text{ and } (w_n^{(x_i)}) \text{ is residually in } U \right\}$ then $\varphi^{-1}(U^*) = \left\{ \hat{G}_x : x \in U \right\} \cup \left\{ \hat{G} : \left( w_n^{(\hat{g})} \right) \text{ is residually in } U \right\}$. If $\hat{G}_x$ converges to $x$ in $U$, then there is an open set $H \in \hat{G}_x$ such that $H \subset U$. This implies that $(X - U) \notin \hat{G}_x$; i.e., $\hat{G}_x \in \kappa X - (X - U)^*$. If $\left( w_n^{(\hat{g})} \right)$ is eventually in $U$, since $\left( w_n^{(\hat{g})} \right)$ is the net based on $\hat{G}$, the corollary 2.1. implies that $U$ is in $\hat{G}$, thus $\hat{G}$ is eventually $U$; i.e., there exists an $G$ in $\hat{G}$ such that $G \subset U$. This implies again that $(X - U) \notin \hat{G}_x$ and therefore $\hat{G}_x \in \kappa X - (X - U)^*$. Thus $\varphi^{-1}(U^*) \subset \kappa X - (X - U)^*$. For $\kappa X - (X - U)^* \subset \varphi^{-1}(U^*)$, let $\hat{G}$ be open ultrafilter in $\kappa X - (X - U)^*$, then $(X - U) \notin \hat{G}$. This implies that there exists an $G_0 \in \hat{G}$ such that $G_0 \cap (X - U) = \emptyset$; i.e., $G_0 \subset U$. Hence,

a) If $\hat{G}$ converges to a point $x$ in $X$; i.e., $\hat{G} = \hat{G}_x$. Then $x$ is in $G$ for all $G$ in $\hat{G}_x$ and thus $x \in G_0 \subset U$. This implies that $\hat{G} = \hat{G}_x$ in $\varphi^{-1}(U^*)$

b) If $\hat{G}$ does not converge in $X$, $G_0 \subset U$ implies that $\hat{G}$ is eventually in $U$, i.e., $U \in \hat{G}$. So, the net $\left( w_n^{(\hat{g})} \right)$ based on $\hat{G}$ is eventually in $U$; i.e., $\hat{G}$ is in $\varphi^{-1}(U^*)$. Thus $\varphi^{-1}(U^*) = \kappa X - (X - U)^*$ is open in $\kappa X$. Hence $\varphi$ is continuous. Since $\varphi^{-1}(U^* \cap \varphi(\kappa X)) = \varphi^{-1}(U^*) \cap \varphi^{-1}(\varphi(\kappa X)) = (\kappa X - (X - U)^*) \cap \kappa X = \kappa X - (X - U)^*$, thus $\varphi(\kappa X - (X - U)^*) = U^* \cap \varphi(\kappa X)$ is an open in $\varphi(\kappa X)$ for
any open set $\kappa X - (X - U)^*$ in $\kappa X$. Hence, $\varphi^{-1}$ is continuous on $\varphi(\kappa X)$. Therefore, $\varphi$ is an embedding of $\kappa X$ into $X^*$.

Theorem 3.2. The Wallman compactification $(\gamma X, h)$ of $X$ can be embedded into the Fan-Gottesman compactification of $X$, if $X$ is $T_3$.

Proof. It is defined a map from $\gamma X$ into $\kappa X$ to proof the theorem. It is considered base defined by closed ultrafilter as a normal base. Let $\vartheta : \gamma X \to \kappa X$ be defined by setting that $\vartheta(F_x) = \hat{G}_x$ such that $x$ contained in $\hat{G}_x$, if $F_x$ is the closed ultrafilter converging to $x$ in $X$. $\vartheta(F) = \left(\left(\hat{F}_n\right)^*\right)^*$, $\left(\hat{F}_n\right)^*$ is the net based on open filter $\hat{G}$, if $F$ is the closed ultrafilter that does not converging in $X$.

$$\vartheta = \begin{cases} 
\hat{G}_x, & \text{if } F_x \text{ is the closed ultrafilter converging to } x \text{ in } X \\
\left(\left(\hat{F}_n\right)^*\right)^*, & \text{if } F \text{ is the closed ultrafilter that does not converging in } X 
\end{cases}$$

It must be shown that $\vartheta$ is an embedding between $\gamma X$ and $\kappa X$. If $F_x$ and $F_y$ are two closed ultra filters converging to $x$ and $y$, respectively, and $F_x \neq F_y$. Then $\vartheta(F_x) = \hat{G}_x$ and $\vartheta(F_y) = \hat{G}_y$. Then $\hat{G}_x \neq \hat{G}_y$. If $F_1, F_2$ are two closed ultra filters that don’t converge in $X$ and $F_1 \neq F_2$, then $\vartheta(F_1) = \left(\left(\hat{F}_n^{(F_1)}\right)^*\right)^*$ and $\vartheta(F_2) = \left(\left(\hat{F}_n^{(F_2)}\right)^*\right)^*$. Since $F_1, F_2$ are two different open ultra filters, the nets $\left(\left(\hat{F}_n^{(F_1)}\right)^*\right)^*$ and $\left(\left(\hat{F}_n^{(F_2)}\right)^*\right)^*$ based on $F_1$ and $F_2$, respectively, are different. Then $\left(\left(\hat{F}_n^{(F_1)}\right)^*\right)^* \neq \left(\left(\hat{F}_n^{(F_2)}\right)^*\right)^*$ in $Y$. If $F_x$ is a closed ultrafilters converging to $x$ in $X$ and $F$ is a closed ultrafilters that does not converge in $X$, then $F_x \neq F$. Since $\varphi(F_x) = \hat{G}_x$, $x$ contained in $\hat{G}_x$, $\varphi(F) = \left(\left(\hat{F}_n\right)^*\right)^*$, so $\varphi(F_x) \neq \varphi(F)$. Therefore, $\vartheta$ is one to one.

Let $U^*$ be open set in $\beta$; i.e.,

$$U^* = \{ \hat{G} \in \kappa X : U \subset \text{cl}_X U \subset V, \text{ } V \text{ is open in } X \text{ and } V \in \hat{G} \}$$
then \( \vartheta^{-1}(U^\bullet) = \{ F_x : x \in U \} \cup \left\{ F : \left( w_n^{(F)} \right) \text{ is eventually in } U \right\} \). If \( F_x \) converges to \( x \) in \( U \), then, there is an \( F \) in \( F_x \) such that \( F \subset U \). If \( \left( w_n^{(F)} \right) \) is eventually in \( U \), since \( \left( w_n^{(F)} \right) \) is the net based on open filter \( \mathcal{G} \) induced by \( F \), \( F \) is eventually in \( U \). Hence, if \( F_x \) converges to \( x \) in \( X \), it is clearly seen that \( F_x = F \). If \( F_x \) does not converge to \( x \) in \( X \), then, \( U \in \mathcal{G} \). So, the net \( \left( w_n^{(F)} \right) \) based on \( G \) is eventually in \( U \). If \( \left( w_n^{(F)} \right) \) is eventually in \( \vartheta^{-1}(U^\bullet) \). Thus \( \vartheta^{-1}(U^\bullet) = \gamma X - (X - U) \) is an open in \( \gamma X \). Hence \( \vartheta \) is continuous.

Since \( \vartheta^{-1}(U^\bullet \cap \vartheta(\gamma X)) = \vartheta^{-1}(U^\bullet) \cap \vartheta^{-1}(\gamma X) = \left( \vartheta^{-1}(\gamma X - (X - U)) \cap \gamma X = \gamma X - (X - U) \right) \cap \gamma X = \gamma X - (X - U) \), \( \vartheta \left( \vartheta^{-1}(\gamma X - (X - U)) \right) = U^\bullet \cap \vartheta(\gamma X) \) is open in \( \vartheta(\gamma X) \) for any open set \( \gamma X - (X - U) \) in \( \gamma X \). Hence, \( \vartheta^{-1} \) is continuous on \( \vartheta(\gamma X) \). Therefore \( \vartheta \) is an embedding of \( \gamma X \) into \( \kappa X \). \( \square \)

4. \( T_0 \)-compactification and H-spectral space

Let \( R \) be a commutative ring with identity. Spectrum or prime spectrum of \( R \), denoted \( \text{Spec}(R) \), is the set of prime ideals of \( R \). The topology on \( \text{Spec}(R) \) defined by closed set \( Z(I) = \{ C \in \text{Spec}(R) : I \subseteq C \} \) for ideals \( I \) of \( R \) is called Zariski topology on \( \text{Spec}(R) \).

By definition, the closure in the Zariski topology of the singleton set \( \{ P \} \) in \( \text{Spec}(R) \) consist of all prime ideals of \( R \) contain \( P \). In particular, a point \( P \) in \( \text{Spec}(R) \) is closed in the Zariski topology if and only if the prime ideal \( P \) is not contained in any other prime ideals of \( R \), i.e., if and only if \( P \) is a maximal ideal \([3]\).

A topological space is called spectral if it is homeomorphic to the prime spectrum or a ring equipped with Zariski topology. M. Hochster \([8]\) has characterized spectral spaces as follows:

A space \( X \) is spectral if and only if the following axioms hold:

1. Every nonempty irreducible closed subset of \( X \) is the closure of a unique point (that is, sober)
(2) $X$ is compact;
(3) The compact open sets form a basis of $X$;
(4) The family of compact open sets of $X$ is closed under finite intersections.

H. Herrlich has introduced the following construction [7]

Let $X$ be a $T_0$-space. Consider the set $\Gamma(X)$ of all filters $F$ on $X$ that satisfy the following two conditions:

(1) $F$ does not converge in $X$.
(2) Every finite open cover of $X$ contains some member of $F$

Let $\Omega(X)$ is the set of minimal elements of $\Gamma(X)$ and define:

$\mathbf{a}: \ X^*_w = X \cup \Omega(X)$.

$\mathbf{b}: \ A^*_w = A \cup \{ F : F \in \Omega(X) \text{ and } A \in F \}$

$\beta_w = \{ A^*_w : A \text{ open in } X \}$ is a base for a topology $\tau^*_w$ on $X^*_w$. $(X^*_w, \tau^*_w)$ is compact and called $T_0$-compactification of $X$ and denoted by $\beta_w X$.

Also, the following properties hold:

(1) If $\beta_w X$ is sober, then $X$ is sober.
(2) If $\beta_w X$ is spectral, then $X$ is sober.
(3) If $\beta_w X$ is normal, then $X$ is normal
(4) If $X$ is normal, then for each distinct elements $H$ and $G$ of $\beta_w X$ there exist two disjoint open sets $U$ and $V$ of $X$ such that $H \in U^*_w$ and $G \in V^*_w$.
(5) If $X$ is normal sober space, then $\beta_w X$ is sober.

**Definition 4.1.** A subset $N$ of a space $X$ is called nearly closed in $X$, if there exist a finite subset $\delta_x$ of $\delta$ and neighborhood $V_x$ of $x$ with $(V_x \cap N) \subseteq \bigcup_{\delta_x, \delta_x \in \delta_x} \delta_x$ for every open cover $\delta$ of $N$ and every point $x$ of $X$.

The specialization order of a topological space $X$ is defined by $x \leq y$ if and only if $y \in \{ x \}$. We denoted by $(x \uparrow) = \{ y \in X : x \leq y \}$ and $(\downarrow x) = \{ y \in X : y \leq x \}$. 

Proposition 4.1. Let \( X \) be a \( T_0 \)-space such that \( (\downarrow x) \cap (\downarrow F) = \emptyset \) for each \( x \in X \) and each \( F \notin (x \uparrow) \cap \Omega(X) \). If \( X \) is H-spectral space, then the following properties hold:

1. If \( C \) is compact open set of \( \beta w X \), then \( C \cap X \) is nearly closed set of \( X \).
2. The nearly closed and open sets form a basis of \( X \).
3. If \( U, V \) are two open sets such that \( U \cup V = X \). Then there exists an open nearly closed set \( N \) of \( X \) such that \( N \subseteq U \) and \( N \cup V = X \).

Remark 1. If \( X \) is \( T_1 \)-space, then \( (\downarrow x) \cap (\downarrow F) = \emptyset \) for each \( x \in X \) and each \( F \notin (x \uparrow) \).

Let us cite [2, 6] for detailed information about this topic.

Karim Belaid at al. [1] have characterized A-spectral spaces (that is; one point compactification of \( X \) is spectral space) and he give some properties of H-spectral spaces (that is; \( T_0 \)-compactification of \( X \) is spectral space) and defined W-spectral spaces (that is; Wallman compactification of \( X \) is spectral space) and characterized of W-spectral spaces [2].

Definition 4.2. Let \( X \) be a \( T_3 \) space. If its Fan-Gottesman compactification is spectral, it is called F-spectral space [4].

Theorem 4.1. Let \( X \) be a \( T_3 \) space. Then \( X \) is an F-spectral if and only if there exists a clopen set \( U \) such that \( G \subseteq U \) and \( H \cap U = \emptyset \) for each disjoint open set \( G \) and \( H \) of \( X \).

Proof. (\( \Rightarrow \)) If \( G \cap H = \emptyset \), then \( (X - G) \cup (X - H) = X \). By 4.1. Proposition and Remark, there is an open nearly closed set \( K \) such that \( K \subseteq (X - G) \) and \( K \cup (X - G) = X \). Therefore \( G \subseteq (X - K) \) and \( G \cap (X - K) = \emptyset \). On the other hand \( \kappa X \) and \( X \) are Hausdorff, we get that \( (X - K) \) is clopen.
(⇐) Let $\gamma = \{ U^* : U \text{ clopen set of } X \}$. Let $V$ be an open set of $X$ and $x \in V^*$. If $x \in V$, then $\{ x \}$ is closed. Because $X$ be a $T_3$ space, $X$ be a $T_1$ and regular. Hence there exists a clopen set $U$ such that $\{ x \} \subseteq U \subseteq V$. Thus $U^*$ is clopen neighborhood of $x$ such that $U^* \subseteq V^*x = \mathcal{R} \in V^* \cap \Omega(X)$, where $\Omega(X)$ is the set of minimal elements of all filters on $X$. For $\varphi \in \kappa X - U^*$, there exist $G \in \mathcal{R}$ and $H \in \varphi$ such that $G \cap H = \emptyset$. Thus there exists a clopen set $U_\varphi$ of $X$ such that $G \subseteq U$ and $G \in (X - U_\varphi)$. Hence $\{(X - U_\varphi)^* : \varphi \in \kappa X - V^* \}$ is an open cover of $\kappa X - V^*$. Since $\kappa X - V^*$ is compact, there is a finite collection $\{(X - U_\varphi)^* : \varphi \in I \}$ such that $\kappa X - V^* = \bigcup \{ (X - U_\varphi)^* : \varphi \in I \}$. Let $U_{\mathcal{R}} = \bigcap \{ U_\varphi : \varphi \in I \}$. It is immediate that $U_{\mathcal{R}}^*$ is a clopen neighborhood of $\mathcal{R}$ such that $U_{\mathcal{R}}^* \subseteq V^*$. Therefore, $\gamma$ is bases of $\kappa X$. Since each element of $\gamma$ is clopen, $\gamma$ is basis of compact sets closed under finite intersection. Every nonempty irreducible close subset of $\kappa X$ is closure of unique point (that is sober). Thus $\kappa X$ is spectral.

**Conclusion 4.1.** Let $X$ be a $T_3$ space. If $X$ is an F-spectral, then $X$ is a W-spectral.

**Proof.** Since $X$ is a $T_3$ space. The Wallman compactification $(\gamma X, h)$ of $X$ can be embedded into the Fan-Gottesman compactification of $X$ from Theorem 2.1. On the other hand, for each disjoint open set $G$ and $H$ of $X$, there exists a clopen set $U$ such that $G \subseteq U$ and $H \cap U = \emptyset$, since $X$ is an F-spectral. Then $X$ is a W-spectral from definition of relative topology and 2.4 Theorem in [2].

**References**


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