Abstract. In this paper, some properties of \(\alpha grw\)-continuous functions are discussed and the notion of \(\alpha grw\)-closed graph is introduced.

1. Introduction

The Pasting lemma for continuous functions has applications in algebraic topology. The continuous functions defined on closed sets of a locally finite covering of a topological space can be pasted to form a continuous function on the whole space. Several mathematicians have established pasting lemmas for some stronger and weaker forms of continuous functions. In this paper pasting lemma for \(\alpha grw\)-continuous functions is proved and also \(\alpha grw\)-closed graph functions are introduced in topological spaces.

Throughout this paper, the space \((X, \tau)\) (or simply \(X\)) always means a topological space on which no separation axioms are assumed unless explicitly stated. For a subset \(A\) of a space \(X\), \(cl(A)\), int\((A)\) and \(X - A\) (or \(A^c\)) denote the closure of \(A\), the interior of \(A\) and the complement of \(A\) in \(X\) respectively.

2. Preliminaries

Definition 2.1. A subset \(A\) of a topological space \((X, \tau)\) is called

(1) regular open [12] if \(A = int(cl(A))\) and regular closed if \(A = cl(int(A))\).
(2) pre-open [7] if \( A \subseteq \text{int}(cl(A)) \) and pre-closed if \( \text{cl}(\text{int}(A)) \subseteq A \)
(3) \( \beta \)-open [1] if \( A \subseteq \text{cl}(\text{int}(A)) \) and \( \beta \)-closed if \( \text{int}(\text{cl}(A)) \subseteq A \).
(4) \( \alpha \)-open [8] if \( A \subseteq \text{int}(\text{cl}(A)) \) and \( \alpha \)-closed [6] if \( \text{cl}(\text{int}(A)) \subseteq A \).

**Definition 2.2.** [3] A subset \( A \) of a space \( (X, \tau) \) is called regular semi-open if there is a regular open set \( U \) such that \( U \subseteq A \subseteq \text{cl}(U) \). The family of all regular semi-open sets of \( X \) is denoted by \( \text{RSO}(X) \).

**Definition 2.3.** [9] A subset \( A \) of a topological space \( (X, \tau) \) is said to be \( \text{agrw}\)-closed if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is regular semi-open.

A subset \( A \) of a topological space \( (X, \tau) \) is said to be \( \text{agrw}\)-open [11] if \( A^c \) is \( \text{agrw}\)-closed.

The set of all \( \text{agrw}\)-closed sets and \( \text{agrw}\)-open sets are denoted by \( \text{agrwC}(X) \) and \( \text{agrwO}(X) \) respectively.

**Definition 2.4.** [6] A function \( f : (X, \tau) \to (Y, \sigma) \) is said to be \( \alpha \)-closed if \( f(U) \) is an \( \alpha \)-closed set of \( (Y, \sigma) \) for every closed set \( U \) of \( (X, \tau) \).

**Definition 2.5.** [10] A function \( f : (X, \tau) \to (Y, \sigma) \) is said to be \( \text{agrw}\)-continuous if \( f^{-1}(V) \) is an \( \text{agrw}\)-closed set of \( (X, \tau) \) for every closed set \( V \) of \( (Y, \sigma) \).

**Definition 2.6.** [10] A function \( f : (X, \tau) \to (Y, \sigma) \) is said to be \( \text{agrw}\)-irresolute if \( f^{-1}(V) \) is an \( \text{agrw}\)-closed set of \( (X, \tau) \) for every \( \text{agrw}\)-closed set \( V \) of \( (Y, \sigma) \).

**Definition 2.7.** [8] A topological space \( (X, \tau) \) is an \( \alpha \)-space if every \( \alpha \)-closed subset of \( (X, \tau) \) is closed in \( (X, \tau) \).

**Definition 2.8.** [5] A function \( f : (X, \tau) \to (Y, \sigma) \) has an \( \alpha \)-closed graph if for each \( (x, y) \notin G(f) \), there exists an \( \alpha \)-open set \( U \) and an open set \( V \) containing \( x \) and \( y \) respectively such that \( (U \times \text{cl}(V)) \cap G(f) = \emptyset \).
Lemma 2.1. [2] Let $A \subseteq Y \subseteq X$, where $X$ is a topological space and $Y$ is open subspace of $X$. If $A \in RSO(X)$, then $A \in RSO(Y)$.

3. $agrw$-CONTINUOUS FUNCTIONS

Definition 3.1. A function $f : (X, \tau) \to (Y, \sigma)$ is called regular semi-open* (resp. regular semi-closed*) if $f(V)$ is regular semi-open (resp. regular semi-closed) in $(Y, \sigma)$ for every regular semi-open (resp. regular semi-closed) set $V$ in $(X, \tau)$.

Definition 3.2. A function $f : (X, \tau) \to (Y, \sigma)$ is called regular semi-irresolute if $f^{-1}(V)$ is regular semi-open in $(X, \tau)$ for every regular semi-open $V$ in $(Y, \sigma)$.

Proposition 3.1. If $A$ is $agrw$-closed in a $\alpha$-space $(X, \tau)$ and if $f : (X, \tau) \to (Y, \sigma)$ is regular semi-irresolute and $\alpha$-closed, then $f(A)$ is $agrw$-closed in $(Y, \sigma)$.

Proof. Let $U$ be any regular semi-open in $(Y, \sigma)$ such that $f(A) \subseteq U$. Then $A \subseteq f^{-1}(U)$ and by assumption, $\alpha cl(A) \subseteq f^{-1}(U)$. This implies $f(\alpha cl(A)) \subseteq U$ and $f(\alpha cl(A))$ is $\alpha$-closed. Now, $\alpha cl(f(A)) \subseteq \alpha cl(f(\alpha cl(A))) = f(\alpha cl(A)) \subseteq U$. Therefore $\alpha cl(f(A)) \subseteq U$ and hence $f(A)$ is $agrw$-closed in $(Y, \sigma)$.

Remark 1. The following examples show that no assumption of the above proposition can be removed.

Example 3.1. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$, $Y = \{p, q, r\}$ and $\sigma = \{\emptyset, \{p\}, \{r\}, \{p, r\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = p$, $f(b) = f(c) = r$ and $f(d) = q$. Then the function $f$ is regular semi-irresolute and $\alpha$-closed but $A = \{a\}$ is not an $agrw$-closed in a $\alpha$-space $(X, \tau)$ and so $f(A)$ is not an $agrw$-closed set in $(Y, \sigma)$.

Example 3.2. In Example 3.1, let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = f(c) = r, f(b) = p$ and $f(d) = q$. Then $A = \{a, c\}$ is $agrw$-closed, $f$ is $\alpha$-closed and $X$ is
\( \alpha \)-space but \( f \) is not regular semi-irresolute and so \( f(A) \) is not an \( \alpha gw \)-closed set in \((Y, \sigma)\).

**Example 3.3.** In Example 3.1, let \( f : (X, \tau) \to (Y, \sigma) \) be defined by \( f(a) = f(d) = p \) and \( f(b) = f(c) = r \). Then \( A = \{a, d\} \) is \( \alpha gw \)-closed, \( f \) is regular semi-irresolute and \( X \) is \( \alpha \)-space but \( f \) is not \( \alpha \)-closed and so \( f(A) \) is not an \( \alpha gw \)-closed set in \((Y, \sigma)\).

**Example 3.4.** Let \( X = \{a, b, c, d\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \), \( Y = \{p, q, r\} \) and \( \sigma = \{\emptyset, \{p\}, \{r\}, \{p, r\}, Y\} \). Let \( f : (X, \tau) \to (Y, \sigma) \) be defined by \( f(a) = p, f(b) = f(d) = r \) and \( f(c) = q \). Then the function \( f \) is regular semi-irresolute, \( f \) is \( \alpha \)-closed and \( A = \{d\} \) is \( \alpha gw \)-closed but \( X \) is not \( \alpha \)-space and so \( f(A) \) is not an \( \alpha gw \)-closed set in \((Y, \sigma)\).

**Theorem 3.1.** Let \( f \) be an \( \alpha gw \)-continuous and regular semi-closed* function from a space \((X, \tau)\) to an \( \alpha \)-space \((Y, \sigma)\). Then \( f \) is an \( \alpha gw \)-irresolute function.

**Proof.** Let \( A \) be an \( \alpha gw \)-open subset in \((Y, \sigma)\) and let \( F \) be any regular semi-closed set in \((X, \tau)\) such that \( F \subseteq f^{-1}(A) \). Then \( f(F) \subseteq A \). Since \( f \) is regular semi-closed*, \( f(F) \) is regular semi-closed. Therefore \( f(F) \subseteq \alpha int(A) \) by Theorem 3.1 [11] and so \( F \subseteq f^{-1}(\alpha int(A)) \). Since \( f \) is \( \alpha gw \)-continuous and \( Y \) is an \( \alpha \)-space, \( f^{-1}(\alpha int(A)) \) is \( \alpha gw \)-open in \((X, \tau)\). Thus \( F \subseteq \alpha int(f^{-1}(\alpha int(A))) \subseteq \alpha int(f^{-1}(A)) \) and so \( f^{-1}(A) \) is \( \alpha gw \)-open in \((X, \tau)\) by Theorem 3.1 [11]. The proof is similar for \( \alpha gw \)-closed set.

Remark 2. The following examples show that no assumption of the above theorem can be removed.

**Example 3.5.** Let \( X = \{a, b, c, d\}, \tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\} \), \( Y = \{p, q, r\} \) and \( \sigma = \{\emptyset, \{q\}, \{p, q\}, \{q, r\}, Y\} \). Let \( f : (X, \tau) \to (Y, \sigma) \) be defined by \( f(a) = f(b) = r, f(c) = q \) and \( f(d) = p \). Then the function \( f \) is \( \alpha gw \)-continuous and \( Y \) is \( \alpha \)-space but \( f \) is not regular semi-closed* and so \( f \) is not \( \alpha gw \)-irresolute.
Example 3.6. Let $X = Y = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an identity map. Then the function $f$ is $\agrw$-continuous and regular semi-closed but $Y$ is not an $\alpha$-space and so $f$ is not $\agrw$-irresolute.

Example 3.7. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$, $Y = \{p, q, r\}$ and $\sigma = \{\emptyset, \{p\}, \{r\}, \{p, r\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = f(b) = p, f(b) = r$ and $f(c) = q$. Then the function $f$ is regular semi-closed but $Y$ is $\alpha$-space but $f$ is not $\agrw$-continuous and so $f$ is not $\agrw$-irresolute.

Corollary 3.1. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\agrw$-continuous and regular semi-closed and if $A$ is $\agrw$-closed (or $\agrw$-open) subset of an $\alpha$-space $(Y, \sigma)$, then $f^{-1}(A)$ is $\agrw$-closed (or $\agrw$-open) in $(X, \tau)$.

Corollary 3.2. Let $(X, \tau), (Z, \eta)$ be a topological spaces and $(Y, \sigma)$ be an $\alpha$-space. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\agrw$-continuous and regular semi-closed and $g : (Y, \sigma) \rightarrow (Z, \eta)$ is $\agrw$-continuous then $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is $\agrw$-continuous.

Proof. Let $F$ be any closed set in $(Z, \eta)$. Since $g$ is $\agrw$-continuous, $g^{-1}(F)$ is $\agrw$-closed. By assumption and by Theorem 3.1, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is $\agrw$-closed in $(X, \tau)$ and so $g \circ f$ is $\agrw$-continuous.

Proposition 3.2. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\agrw$-continuous then for each point $x$ in $X$ and each open set $V$ in $Y$ with $f(x) \in V$, there is an $\agrw$-open set $U$ in $X$ such that $x \in U$ and $f(U) \subseteq V$.

Proof. Let $V$ be an open set in $(Y, \sigma)$ and let $f(x) \in V$. Then $x \in f^{-1}(V) \in \agrw O(X)$, since $f$ is $\agrw$-continuous. Let $U = f^{-1}(V)$. Then $x \in U$ and $f(U) \subseteq V$.

Remark 3. The converse of the above proposition need not be true as seen from the following example.
Example 3.8. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{e\}, \{a, c\}, X\}$, $Y = \{p, q, r, s\}$ and $\sigma = \{\emptyset, \{p\}, \{q\}, \{p, q\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = p$, $f(b) = q$ and $f(c) = r$. Then for each point $x$ in $X$ and each open set $V$ in $Y$ with $f(x) \in V$, there is an $\alpha$-grw-open set $U$ in $X$ such that $x \in U$ and $f(U) \subseteq V$ but $f$ is not $\alpha$-grw-continuous.

The following theorem is the Pasting lemma for $\alpha$-grw-continuous functions.

Theorem 3.2. Let $X = A \cup B$, where $A$ and $B$ are $\alpha$-grw-closed and regular open in $X$. Let $f : (A, \tau_A) \to (Y, \sigma)$ and $g : (B, \tau_B) \to (Y, \sigma)$ be $\alpha$-grw-continuous such that $f(x) = g(x)$ for every $x \in A \cap B$. Then the combination $f \circ g : (X, \tau) \to (Y, \sigma)$ defined by $(f \circ g)(x) = f(x)$ if $x \in A$ and $(f \circ g)(x) = g(x)$ if $x \in B$ is $\alpha$-grw-continuous.

Proof. Let $U$ be any closed set in $Y$. Then $(f \circ g)^{-1}(U) = [(f \circ g)^{-1}(U) \cap A] \cup [(f \circ g)^{-1}(U) \cap B] = f^{-1}(U) \cup g^{-1}(U) = C \cup D$, where $C = f^{-1}(U)$ and $D = g^{-1}(U)$. Since $f$ is $\alpha$-grw-continuous, we have $C$ is $\alpha$-grw-closed in $(A, \tau_A)$ and also since $A$ is $\alpha$-grw-closed and regular open in $X$, $C$ is $\alpha$-grw-closed in $X$, by Proposition 7 [4]. Similarly, $D$ is $\alpha$-grw-closed in $X$ and by Theorem 3.19[9], $(f \circ g)^{-1}(U) = C \cup D$ is $\alpha$-grw-closed in $X$. Hence $f \circ g$ is $\alpha$-grw-continuous.

Definition 3.3. A function $f : (X, \tau) \to (Y, \sigma)$ has an $\alpha$-grw-closed graph if for each $(x, y) \notin G(f)$, there exists an $\alpha$-grw-open set $U$ and an open set $V$ containing $x$ and $y$ respectively such that $(U \times \text{cl}(V)) \cap G(f) = \emptyset$.

Example 3.9. Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a\}, X\}$ and $Y = \{p, q, r\}$ with topology $\sigma = \mathcal{P}(Y)$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = p$, $f(b) = q$ and $f(c) = r$. Then $f$ has an $\alpha$-grw-closed graph.

Proposition 3.3. A function with $\alpha$-closed graph has an $\alpha$-grw-closed graph.
Proof. Let \( f : (X, \tau) \to (Y, \sigma) \) has an \( \alpha \)-closed graph. Then there exists an \( \alpha \)-open set \( U \) and an open set \( V \) containing \( x \) and \( y \) respectively if for each \( (x, y) \in G(f) \) such that \( (U \times \text{cl}(V)) \cap G(f) = \emptyset \). Since every \( \alpha \)-open set is an \( \alpha grw \)-open set[9]. Therefore \( U \) is an \( \alpha \)-open set. Hence \( f \) has an \( \alpha grw \)-closed graph.

Remark 4. The converses of the above proposition need not be true in general. In Example 3.9, the function \( f \) has an \( \alpha grw \)-closed graph but not has an \( \alpha \)-closed graph.

Lemma 3.1. The function \( f : (X, \tau) \to (Y, \sigma) \) has an \( \alpha grw \)-closed graph if and only if for each \( (x, y) \in X \times Y \) such that \( f(x) \neq y \), there exist an \( \alpha grw \)-open set \( U \) and an open set \( V \) containing \( x \) and \( y \) respectively, such that \( f(U) \cap \text{cl}(V) = \emptyset \).

Proof. Necessity. Let for each \( (x, y) \in X \times Y \) such that \( f(x) \neq y \). Then there exist an \( \alpha grw \)-open set \( U \) and an open set \( V \) containing \( x \) and \( y \), respectively, such that \( (U \times \text{cl}(V)) \cap G(f) = \emptyset \), since \( f \) has an \( \alpha grw \)-closed graph. Hence for each \( x \in U \) and \( y \in \text{cl}(V) \) with \( y \neq f(x) \), we have \( f(U) \cap \text{cl}(V) = \emptyset \).

Sufficiency. Let \( (x, y) \notin G(f) \). Then \( y \neq f(x) \) and so there exist an \( \alpha grw \)-open set \( U \) and an open set \( V \) containing \( x \) and \( y \), respectively, such that \( f(U) \cap \text{cl}(V) = \emptyset \). This implies, for each \( x \in U \) and \( y \in \text{cl}(V) \), \( f(x) \neq y \). Therefore \( (U \times \text{cl}(V)) \cap G(f) = \emptyset \). Hence \( f \) has an \( \alpha grw \)-closed graph.

Theorem 3.3. If \( f \) is an \( \alpha grw \)-continuous function from a space \( X \) into a Hausdorff space \( Y \), then \( f \) has an \( \alpha grw \)-closed graph.

Proof. Let \( (x, y) \notin G(f) \). Then \( y \neq f(x) \). Since \( Y \) is Hausdorff space, there exist two disjoint open sets \( V \) and \( W \) such that \( f(x) \in W \) and \( y \in V \). Since \( f \) is \( \alpha grw \)-continuous, there exists an \( \alpha grw \)-open set \( U \) such that \( x \in U \) and \( f(U) \subseteq W \) by Proposition 3.2. Thus \( f(U) \subseteq Y - \text{cl}(V) \). Therefore \( f(U) \cap \text{cl}(V) = \emptyset \) and so \( f \) has an \( \alpha grw \)-closed graph.
Theorem 3.4. If $f$ is a surjective function with an $\alpha grw$-closed graph from a space $X$ onto a space $Y$, then $Y$ is Hausdorff.

Proof. Let $y_1$ and $y_2$ be two distinct points in $Y$. Then there exists a point $x_1 \in X$ such that $f(x_1) = y_1 \neq y_2$. Thus $(x_1, y_2) \notin G(f)$. Since $f$ has an $\alpha grw$-closed graph, there exist an $\alpha grw$-open set $U$ and an open set $V$ containing $x_1$ and $y_2$, respectively, such that $f(U) \cap \text{cl}(V) = \emptyset$ and so $f(x_1) \notin \text{cl}(V)$. Hence $Y$ is Hausdorff.

Proposition 3.4. The space $X$ is Hausdorff if and only if the identity mapping $f : X \to X$ has an $\alpha grw$-closed graph.

Proof. Obvious from Theorem 3.3 and 3.4.

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